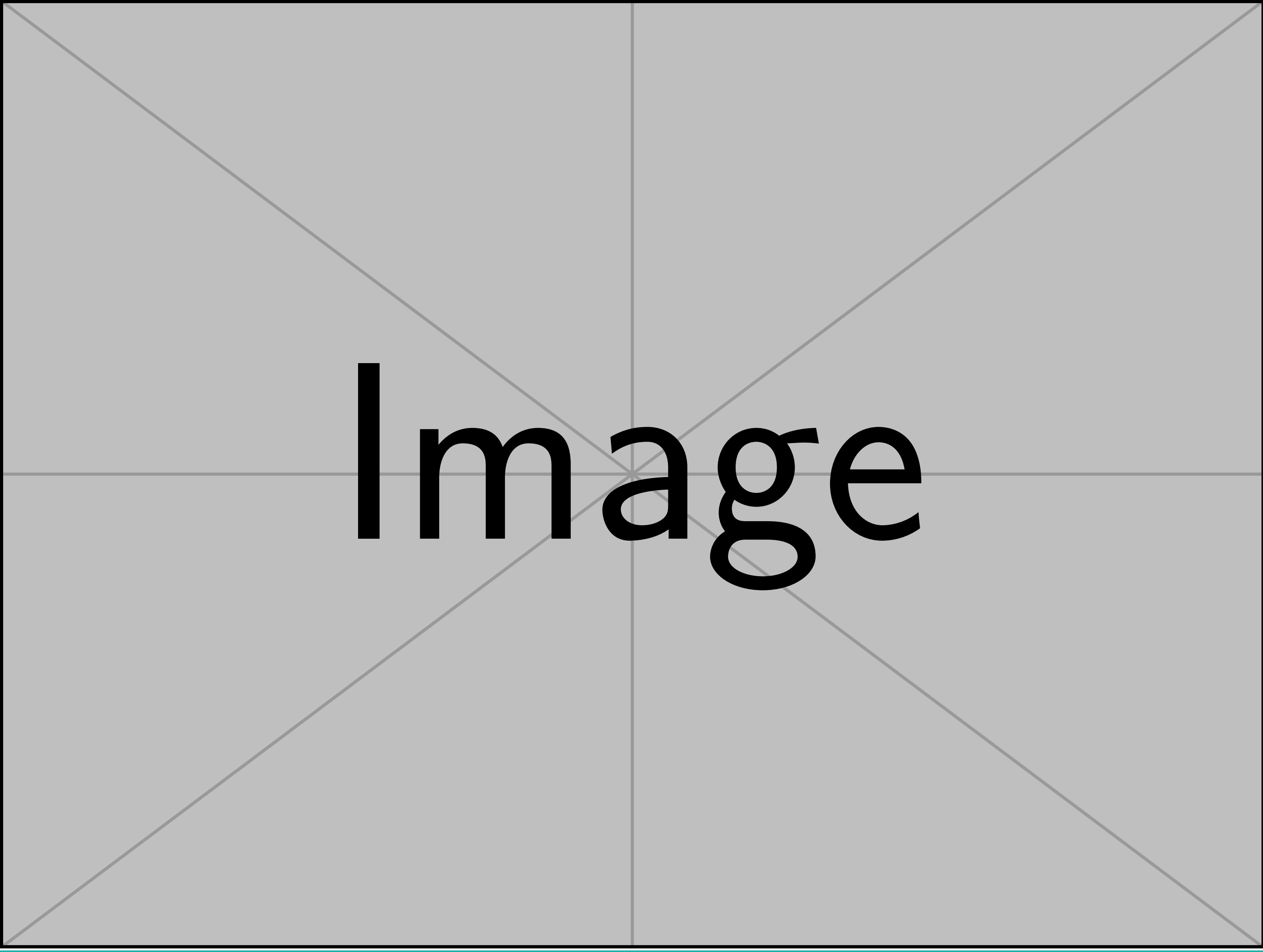

Notes on Mathematics II

Complex & Vector Analysis

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Victory won't come to us unless we go to it.

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Chapter 1 Introduction to Vector Analysis

Introduction

❏ *Exercise*

Chapter 1 Exercise

1. Define the following:
   (a). Scalar
   (b). Vector
   (c). Unite Vector
   (d). Null Vector
   (e). Equal Vector
   (f). Like Vector
   (g). Colinear vectors
   (h). Coplaner vectors
   (i). Scalar multiplication of a vector
   (j). Addition of vectors
2. Write down the following
   (a). Commutative law of vector addition
   (b). Associative law of vector addition
3. Write short note on following topics
   (a). Position of a vector point
4. Find the distance between the points $(-4,-5)$ and $(-1,-1)$.
5. Find the slope of the line $5x-5y=7$.

Chapter 2 Vector Multiplications

Introduction

- *Scalar or Dot Product*
- *Vector or Cross Product*
- *Scalar Triple Product*
- *Vector Triple Product*
- *Scalar Product of Four vectors*
- *Vector Product of Four vectors*
- *Exercise*

2.1 Scalar or Dot Product

Definition 2.1 (Scalar or Dot Product)

The scalar or dot product *of two vectors $\vec{u}$ and $\vec{v}$ produce a scalar, and denoted by $\vec{u}\cdot\vec{v}$ (read: $\vec{u}$ dot $\vec{v}$), is defined as*

$$\vec{u}\cdot\vec{v} = uv\cos\theta, \qquad 0 \le \theta \le \pi, \tag{2.1}$$

where θ is the angle between $\vec{u}$ and $\vec{v}$. ♣

Note *Though $\vec{u}$ and $\vec{v}$ are vectors, $\vec{u}\cdot\vec{v}$ is a scalar.*

Example 2.1 If $\vec{F}$, $\vec{s}$ are force and displacement vectors respectively then their scalar product produce a scalar quantity work w,

$$w = \vec{F}\cdot\vec{s} = Fs\cos\theta.$$

The following proposition applies.

Proposition 2.1

Suppose $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors and m is a scalar. Then the following laws hold:

1. $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$ *Commutative Law*
2. $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$ *Distributive Law*
3. $m\,(\vec{u}\cdot\vec{v}) = (m\vec{u})\cdot\vec{v} = \vec{u}\cdot(m\vec{v}) = (\vec{u}\cdot\vec{v})\,m.$
4. $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1.$
5. $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{i} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{j} = 0.$
6. *If $\vec{u}\cdot\vec{v} = 0$ and $\vec{u}$ and $\vec{v}$ are not null vectors, then $\vec{u}$ and $\vec{v}$ are perpendicular.* ♠

There is a simple formula for $\vec{u}\cdot\vec{v}$ when the unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ are used.

Proposition 2.2

Given $\vec{u} = u_x\hat{i} + u_y\hat{j} + u_z\hat{k}$ and $\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$, then

$$\vec{u}\cdot\vec{v} = u_xv_x + u_yv_y + u_zv_z. \tag{2.2}$$

♠

Corollary 2.1

If $\vec{u} = u_x\hat{i} + u_y\hat{j} + u_z\hat{k}$ then $\vec{u} \cdot \vec{u} = u_x^2 + u_y^2 + u_z^2$.

2.2 Vector or Cross Product

Definition 2.2 (Vector or Cross Product)

The vector or cross product *of two vectors $\vec{u}$ and $\vec{v}$ produce a vector, and denoted by $\vec{u} \times \vec{v}$ (read: $\vec{u}$ cross $\vec{v}$), is defined as*

$$\vec{u} \times \vec{v} = uv \sin\theta\hat{\eta}, \qquad 0 \leq \theta \leq \pi, \tag{2.3}$$

where θ is the angle between $\vec{u}$ and $\vec{v}$, and the direction of the vector $\vec{u} \times \vec{v}$ is denoted by a unite vector $\hat{\eta}$ is perpendicular to the plane of $\vec{u}$, and $\vec{v}$, and is determined by the right-handed system.

Example 2.2 If $\vec{F}$, $\vec{r}$ are force and position vectors respectively then their vector product produce a vector quantity torque $\vec{\tau}$,

$$\vec{\tau} = \vec{r} \times \vec{F} = rF \sin\theta\hat{\eta}.$$

The following proposition applies.

Proposition 2.3

Suppose $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors and m is a scalar. Then the following laws hold:

1. $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$ *Commutative Law fails*
2. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$ *Distributive Law*
3. $m(\vec{u} \times \vec{v}) = (m\vec{u}) \times \vec{v} = \vec{u} \times (m\vec{v}) = (\vec{u} \times \vec{v})\, m$.
4. $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$.
5. $$\begin{aligned} \hat{i} \times \hat{j} &= \hat{k}; \\ \hat{j} \times \hat{k} &= \hat{i}; \\ \hat{j} \times \hat{k} &= \hat{i}; \\ \hat{j} \times \hat{i} &= -\hat{k}; \\ \hat{k} \times \hat{j} &= -\hat{i}; \\ \hat{k} \times \hat{j} &= -\hat{i} \end{aligned}$$
6. *If $\vec{u} \times \vec{v} = 0$ and $\vec{u}$ and $\vec{v}$ are not null vectors, then $\vec{u}$ and $\vec{v}$ are parallel.*
7. *The magnitude of $\vec{u} \times \vec{v}$ is the same as the area of a parallelogram with sides $\vec{u}$ and $\vec{v}$.*

Proposition 2.4

Given $\vec{u} = u_x\hat{i} + u_y\hat{j} + u_z\hat{k}$ and $\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$, then

$$
\begin{aligned}
\vec{u} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} && (2.4) \\
&= (u_y v_z - u_z v_y)\hat{i} + (u_z v_x - u_x v_z)\hat{j} + (u_x v_y - u_y v_x)\hat{k} && (2.5)
\end{aligned}
$$

♠

Corollary 2.2

If $\vec{u} = u_x\hat{i} + u_y\hat{j} + u_z\hat{k}$ then $\vec{u} \times \vec{u} = 0$.

♡

2.3 Scalar Triple Product

Definition 2.3 (Scalar Triple Vector)

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ be three vectors, then the scalar product of $\vec{u}$ with $\vec{v} \times \vec{w}$, (or $\vec{u} \times \vec{v}$ with $\vec{w}$) is called the scalar triple product *(STP) of $\vec{u}$, $\vec{v}$, $\vec{w}$, and written as*

$$[\vec{u}\ \ \vec{v}\ \ \vec{w}] = \vec{u} \cdot \vec{v} \times \vec{w} = \vec{u} \times \vec{v} \cdot \vec{w}.$$

.

♣

When $\vec{u}$, $\vec{v}$ and $\vec{w}$ can be expressed with the unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ then we can get the following proposition.

Proposition 2.5

Given $\vec{u} = u_x\hat{i} + u_y\hat{j} + u_z\hat{k}$, $\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$, and $\vec{w} = w_x\hat{i} + w_y\hat{j} + w_z\hat{k}$ then

$$
\begin{aligned}
\vec{u} \times \vec{v} \cdot \vec{w} &= \left[(u_y v_z - u_z v_y)\hat{i} + (u_z v_x - u_x v_z)\hat{j} + (u_x v_y - u_y v_x)\hat{k}\right] \cdot (w_x\hat{i} + w_y\hat{j} + w_z\hat{k}), \\
&= (u_y v_z - u_z v_y)w_x + (u_z v_x - u_x v_z)w_y + (u_x v_y - u_y v_x)w_z, && (2.6) \\
&= \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}. && (2.7)
\end{aligned}
$$

♠

2.3.1 Properties of STP

Proposition 2.6

If θ is acute, then the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ for a right handed system of vectors,

$$
\begin{aligned}
& \vec{u} \times \vec{v} \cdot \vec{w} = \vec{v} \times \vec{w} \cdot \vec{u} = \vec{w} \times \vec{u} \cdot \vec{v} && (2.8) \\
= \ & \vec{u} \cdot \vec{v} \times \vec{w} = \vec{v} \cdot \vec{w} \times \vec{u} = \vec{w} \cdot \vec{u} \times \vec{v}, \\
= \ & -\vec{v} \times \vec{u} \cdot \vec{w} = -\vec{w} \times \vec{v} \cdot \vec{u} = -\vec{u} \times \vec{w} \cdot \vec{v} \\
= \ & -\vec{v} \cdot \vec{u} \times \vec{w} = \vec{w} \cdot \vec{v} \times \vec{u} = \vec{u} \cdot \vec{w} \times \vec{v}, && (2.9)
\end{aligned}
$$

or,

$$\begin{aligned} & [\vec{u}\ \vec{v}\ \vec{w}] = [\vec{v}\ \vec{w}\ \vec{u}] = [\vec{w}\ \vec{u}\ \vec{v}] \\ = & -[\vec{v}\ \vec{u}\ \vec{w}] = -[\vec{w}\ \vec{v}\ \vec{u}] = -[\vec{u}\ \vec{w}\ \vec{v}] \end{aligned} \tag{2.10}$$

Proof

$$\begin{aligned} [\vec{u}\ \vec{v}\ \vec{w}] &= \vec{u} \times \vec{v} \cdot \vec{w} \\ &= \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} \qquad \text{[using 2.7]} \\ &= \begin{vmatrix} v_x & v_y & v_z \\ w_x & w_y & w_z \\ u_x & u_y & u_z \end{vmatrix} \qquad \text{[Determinant law]} \\ &= \vec{v} \times \vec{w} \cdot \vec{u} \qquad \text{[using 2.7]} \\ &= [\vec{v}\ \vec{w}\ \vec{u}]. \end{aligned} \tag{2.11}$$

Similarly, can be proved using 2.10

$$[\vec{u}\ \vec{v}\ \vec{w}] = [\vec{v}\ \vec{w}\ \vec{u}] = [\vec{w}\ \vec{u}\ \vec{v}].$$

Again

$$\begin{aligned} [\vec{u}\ \vec{v}\ \vec{w}] &= \vec{u} \times \vec{v} \cdot \vec{w} \\ &= -\vec{v} \times \vec{u} \cdot \vec{w} \qquad \text{[using ??]} \\ &= -[\vec{v}\ \vec{u}\ \vec{w}] \qquad \text{[using 2.7]} \end{aligned}$$

Note

1. *The scalar triple is also called parallelepiped law or box product.*
2. *The sign of the scalar triple product remains unchanged if the cyclic order of the vectors is maintained.*
3. *For every change of cycle order a negative sign is introduced.*
4. *The dot and cross may be changed at will.*

2.3.2 Geometric Interpretation of STP

For $\vec{u} \times \vec{v} = uv \sin\theta \hat{\eta}$, where $\hat{\eta}$ is unite vector perpendicular to the plane of $\vec{u}$, and $\vec{v}$. Now,

$$\begin{aligned} [\vec{u}\ \vec{v}\ \vec{w}] &= \vec{u} \cdot \vec{v} \times \vec{w} \\ &= \vec{u} \cdot vw \sin\theta \hat{\eta} \\ &= u \cos\phi\ \ vw \sin\theta \\ &= \text{perpendicular height} \times \text{area of the base}, \\ &= \text{volume of the parallelepiped.} \end{aligned}$$

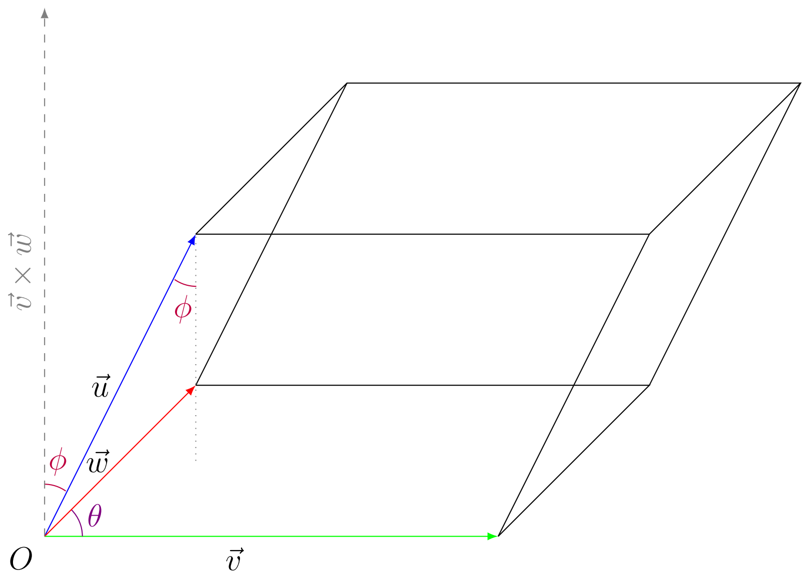


Figure 2.1: A parallelepiped with adjacent vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$.

Corollary 2.3

If the three vectors are coplanar and not parallel, then $[\vec{u} \ \ \vec{v} \ \ \vec{w}] = 0$.

Corollary 2.4

If the two vectors are parallel, then $[\vec{u} \ \ \vec{v} \ \ \vec{w}] = 0$.

Corollary 2.5

If the two vectors are equal, then $[\vec{u} \ \ \vec{u} \ \ \vec{w}] = 0$.

2.4 Vector Triple Product

Definition 2.4 (Vector Triple Vector)

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ be three vectors, then the vector product of $\vec{u}$ with $\vec{v} \times \vec{w}$, (or $\vec{u} \times \vec{v}$ with $\vec{w}$) is called the vector triple product *(VTP) of $\vec{u}$, $\vec{v}$, $\vec{w}$, and written as*

$$\vec{u} \times (\vec{v} \times \vec{w}) \qquad \textit{or}, (\vec{u} \times \vec{v}) \times \vec{w}.$$

.

Proposition 2.7

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are three vectors then

$$\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\, \vec{v} - (\vec{u} \cdot \vec{v})\, \vec{w}. \tag{2.12}$$

Corollary 2.6

$$(\vec{u} \times \vec{v}) \times \vec{w} = (\vec{u} \cdot \vec{w})\, \vec{v} - (\vec{w} \cdot \vec{v})\, \vec{u}. \tag{2.13}$$

Problem 2.1 Find $\vec{a} \times (\vec{b} \times \vec{c})$, where $\vec{a} = -\hat{i} + 2\hat{j} + \hat{k}, \vec{b} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{c} = \hat{i} + 2\hat{j} - 2\hat{k}$,

Solution *We have*

$$\begin{aligned}
\vec{a} \times (\vec{b} \times \vec{c}) &= (\vec{a} \cdot \vec{c})\, \vec{b} - \left(\vec{a} \cdot \vec{b}\right) \vec{c} \\
&= (-1 \cdot 1 + 2 \cdot 2 + 1 \cdot -2)(2\hat{i} + \hat{j} - \hat{k}) - (-1 \cdot 2 + 2 \cdot 1 + 1 \cdot -1)\left(\hat{i} + 2\hat{j} - 2\hat{k}\right) \\
&= (2\hat{i} + \hat{j} - \hat{k}) + \left(\hat{i} + 2\hat{j} - 2\hat{k}\right) \\
&= 3\hat{i} + 3\hat{j} - 3\hat{k}
\end{aligned}$$

2.5 Product of Multiple(Four) vectors

Proposition 2.8

If $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ are four vectors then

$$\left(\vec{a} \times \vec{b}\right) \cdot \left(\vec{c} \times \vec{d}\right) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c}) \tag{2.14}$$

Proof In scalar triple product dot and cross may be interchanged. We have

$$\left(\vec{a} \times \vec{b}\right) \cdot \left(\vec{c} \times \vec{d}\right) = \vec{a} \cdot \vec{b} \times \left(\vec{c} \times \vec{d}\right) = \vec{a} \cdot \left[(\vec{b} \cdot \vec{d})\vec{c} - (\vec{b} \cdot \vec{c})\vec{d}\right] = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{b} \cdot \vec{c})(\vec{a} \cdot \vec{d}).$$

Proposition 2.9

If $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ are four vectors then

$$\left(\vec{a} \times \vec{b}\right) \times \left(\vec{c} \times \vec{d}\right) = \left[\vec{a} \;\; \vec{b} \;\; \vec{d}\right] \vec{c} - \left[\vec{a} \;\; \vec{b} \;\; \vec{c}\right] \vec{d} = \left[\vec{a} \;\; \vec{c} \;\; \vec{d}\right] \vec{b} - \left[\vec{b} \;\; \vec{c} \;\; \vec{d}\right] \vec{a}$$

.

Proof Let $\vec{r} = \vec{a} \times \vec{b}$ then using (2.12) we have,

$$\begin{aligned}
\left(\vec{a} \times \vec{b}\right) \times \left(\vec{c} \times \vec{d}\right) &= \vec{r} \times \left(\vec{c} \times \vec{d}\right) \\
&= \left(\vec{r} \cdot \vec{d}\right) \vec{c} - (\vec{r} \cdot \vec{c})\, \vec{d} \\
&= \left(\left(\vec{a} \times \vec{b}\right) \cdot \vec{d}\right) \vec{c} - \left(\left(\vec{a} \times \vec{b}\right) \cdot \vec{c}\right) \vec{d} \\
\left(\vec{a} \times \vec{b}\right) \times \left(\vec{c} \times \vec{d}\right) &= \left[\vec{a} \;\; \vec{b} \;\; \vec{d}\right] \vec{c} - \left[\vec{a} \;\; \vec{b} \;\; \vec{c}\right] \vec{d} \qquad (2.15)
\end{aligned}$$

Again let $\vec{s} = \vec{c} \times \vec{d}$ then using (2.13) we have,

$$\begin{aligned}
\left(\vec{a} \times \vec{b}\right) \times \left(\vec{c} \times \vec{d}\right) &= \left(\vec{a} \times \vec{b}\right) \times \vec{s} \\
&= (\vec{a} \cdot \vec{s})\, \vec{b} - \left(\vec{b} \cdot \vec{s}\right) \vec{a} \\
\left(\vec{a} \times \vec{b}\right) \times \left(\vec{c} \times \vec{d}\right) &= \left[\vec{a} \;\; \vec{c} \;\; \vec{d}\right] \vec{b} - \left[\vec{b} \;\; \vec{c} \;\; \vec{d}\right] \vec{a} \qquad (2.16)
\end{aligned}$$

From (2.15)-(2.16) we get the following

$$\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)=\left[\vec{a}\;\;\vec{b}\;\;\vec{d}\right]\vec{c}-\left[\vec{a}\;\;\vec{b}\;\;\vec{c}\right]\vec{d}=\left[\vec{a}\;\;\vec{c}\;\;\vec{d}\right]\vec{b}-\left[\vec{b}\;\;\vec{c}\;\;\vec{d}\right]\vec{a}$$

Problem 2.2 Let $\bar{a}$ and $\bar{b}$ be two vectors. Find a vector which is perpendicular to both of them. Show that the volume of the parallelepiped form by the three vectors is

$$a^2b^2-\left(\bar{a}\cdot\bar{b}\right)^2$$

Solution *Let $\vec{c}=\vec{a}\times\vec{b}$ is a vector, which is perpendicular to both of them. Now volume of the parallelepiped is given by*

$$\begin{aligned}\vec{c}\cdot\vec{a}\times\vec{b} &= \left(\vec{a}\times\vec{b}\right)\cdot\left(\vec{a}\times\vec{b}\right)\\ &= \left(\vec{a}\cdot\vec{a}\right)\left(\vec{b}\cdot\vec{b}\right)-\left(\vec{a}\cdot\vec{b}\right)\left(\vec{a}\cdot\vec{b}\right) \qquad [(2.14)]\\ &= a^2b^2-\left(\bar{a}\cdot\bar{b}\right)^2.\end{aligned}$$

Problem 2.3 Let $\vec{a}=(1,2,3)$ and $\vec{b}=(2,-1,3)$ be two vectors. Find a vector $\vec{c}$, which is perpendicular to both $\vec{a}$ and $\vec{b}$. Find the volume of the parallelepiped form by three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Solution *The given vector $\vec{c}$ is given by*

$$\vec{c}=\vec{a}\times\vec{b}=(2\cdot3-(-1)\cdot3)\hat{\imath}+(3\cdot2-1\cdot3)\hat{\jmath}+(1\cdot(-1)-2\cdot2)\hat{k}=9\hat{\imath}+3\hat{\jmath}-5\hat{k}$$

Now the volume of the parallelepiped is as follows

$$\vec{c}\cdot\vec{a}\times\vec{b}=\vec{c}\cdot\vec{c}=c^2=9^2+3^2+(-5)^2=115.$$

Chapter 2 Exercise

1. Define the following
   (a). Scalar product or, dot product
   (b). Vector product or, cross product
   (c). Scalar Triple Product
   (d). Vector Triple Product
2. Write down the following
   (a). Scalar triple product in determinant form.
3. Write short note on following topics
   (a).
4. Short questions
   (a). Let $\vec{a}$ and $\vec{b}$ be are not null vectors, then what is the condition that $\vec{a}$ and $\vec{b}$ are perpendicular.
   (b). Let $\vec{a}=2\hat{\imath}+3\hat{\jmath}$ find two vectors perpendicular to $\vec{a}$.
   (c). Write down the distributive law for the vector products?
   (d). Provide two difference between scalar and vector multiplications of two vectors?
   (e). Given that $\vec{a}=-\hat{\imath}+2\hat{\jmath}+\hat{k}$, $\vec{b}=2\hat{\imath}+\hat{\jmath}-\hat{k}$ and $\vec{c}=\hat{\imath}+2\hat{\jmath}-2\hat{k}$, find $\vec{a}\times\left(\vec{b}\times\vec{c}\right)$, using the determinant.
   (f). Define vector triple product of three vectors?
5. Let $\vec{a}$ and $\vec{b}$ be are not null vectors, then what is the condition that $\vec{a}$ and $\vec{b}$ are perpendicular.

6. Write down the distributive law for the vector products?
7. Provide two difference between scalar and vector multiplications of two vectors?
8. Explain geometric interpretation of scalar triple product.
9. Find $\vec{a} \cdot \vec{b} \times \vec{c}$, for
   (a). $\vec{a} = -\hat{\imath} + 2\hat{\jmath} + \hat{k}, \vec{b} = 2\hat{\imath} + \hat{\jmath} - \hat{k}$ and $\vec{c} = \hat{\imath} + 2\hat{\jmath} - 2\hat{k}$,
10. Let $\vec{a} = (1, 2, 3)$ and $\vec{b} = (2, -1, 3)$ be two vectors. Find a vector $\vec{c}$, which is perpendicular to both $\vec{a}$ and $\vec{b}$.
11. Let $\vec{a} = (1, 2, 3)$ and $\vec{b} = (2, -1, 3)$ be two vectors. Find a unite vector $\hat{c}$, which is perpendicular to both $\vec{a}$ and $\vec{b}$.
12. Using vector triple product show that
$$\bar{a} \times \left(\bar{b} \times \bar{c}\right) + \bar{b} \times (\bar{c} \times \bar{a}) + \bar{c} \times \left(\bar{a} \times \bar{b}\right) \equiv 0$$
13. Find $\vec{a} \times (\vec{b} \times \vec{c})$, for
   (a). $\vec{a} = -\hat{\imath} + 2\hat{\jmath} + \hat{k}, \vec{b} = 2\hat{\imath} + \hat{\jmath} - \hat{k}$ and $\vec{c} = \hat{\imath} + 2\hat{\jmath} - 2\hat{k}$,
14. Let $\bar{a}$ and $\bar{b}$ be two vectors. Find a vector which is perpendicular to both of them. Show that the volume of the parallelepiped form by the three vectors is
$$a^2b^2 - \left(\bar{a} \cdot \bar{b}\right)^2.$$
15. Let $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ are vectors. Prove that
$$\left(\vec{a} \times \vec{b}\right) \times \left(\vec{c} \times \vec{d}\right) = \left[\vec{a} \;\; \vec{b} \;\; \vec{d}\right] \vec{c} - \left[\vec{a} \;\; \vec{b} \;\; \vec{c}\right] \vec{d} = \left[\vec{a} \;\; \vec{c} \;\; \vec{d}\right] \vec{b} - \left[\vec{b} \;\; \vec{c} \;\; \vec{d}\right] \vec{a}.$$
16. Let $\vec{a} = (1, 2, 3)$ and $\vec{b} = (2, -1, 3)$ be two vectors. Find a vector $\vec{c}$, which is perpendicular to both $\vec{a}$ and $\vec{b}$. Find the volume of the parallelepiped form by three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Chapter 3 Differentiation of Vectors

Introduction

- *Derivatives of Vector-Valued Functions*
- *Point Function*
- *Differentiation of Vectors*
- *Gradient*
- *Divergence*
- *Curl*
- *Exercise*

3.1 Ordinary Derivatives of Vector-Valued Functions

Let the position vector $\vec{r}(t)$ joining the origin O of a coordinate system and any point (x, y, z). Then

$$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k} \tag{3.1}$$

and the specification of the vector function $\vec{r}(t)$ defines x, y, and z as functions of t.

3.1.1 Some Rules on Derivatives of Vector-Valued Functions

Proposition 3.1

Let $\vec{u}, \vec{v}$, and $\vec{w}$ are differentiable vector functions of a scalar t, and ϕ is a differentiable scalar function of t. Then the following laws hold:

$$\frac{d}{dt}(\vec{u} \pm \vec{v}) = \frac{d\vec{u}}{dt} \pm \frac{d\vec{v}}{dt} \tag{3.2}$$

$$\frac{d}{dt}(\vec{u} \cdot \vec{v}) = \vec{u} \cdot \frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt} \cdot \vec{v} \tag{3.3}$$

$$\frac{d}{dt}(\vec{u} \times \vec{v}) = \vec{u} \times \frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt} \times \vec{v} \tag{3.4}$$

$$\frac{d}{dt}(\phi\vec{u}) = \phi\frac{d\vec{u}}{dt} + \frac{d\phi}{dt}\vec{u} \tag{3.5}$$

$$\frac{d}{dt}(\vec{u} \cdot \vec{v} \times \vec{w}) = \vec{u} \cdot \vec{v} \times \frac{d\vec{w}}{dt} + \vec{u} \cdot \frac{d\vec{v}}{dt} \times \vec{w} + \frac{d\vec{u}}{dt} \cdot \vec{v} \times \vec{w} \tag{3.6}$$

$$\frac{d}{dt}\{\vec{u} \times (\vec{v} \times \vec{w})\} = \vec{u} \times \left(\vec{v} \times \frac{d\vec{w}}{dt}\right) + \vec{u} \times \left(\frac{d\vec{v}}{dt} \times \vec{w}\right) + \frac{d\vec{u}}{dt} \times (\vec{v} \times \vec{w}) \tag{3.7}$$

♠

Proposition 3.2

Let $\vec{u}$ and $\vec{v}$ are vector function of x, y, z. Then the following laws hold:

$$\frac{\partial}{\partial t}(\vec{u} \pm \vec{v}) = \frac{\partial \vec{u}}{\partial t} \pm \frac{\partial \vec{v}}{\partial t} \tag{3.8}$$

$$\frac{\partial}{\partial t}(\vec{u} \cdot \vec{v}) = \vec{u} \cdot \frac{\partial \vec{v}}{\partial t} + \frac{\partial \vec{u}}{\partial t} \cdot \vec{v} \tag{3.9}$$

$$\frac{\partial}{\partial t}(\vec{u} \times \vec{v}) = \vec{u} \times \frac{\partial \vec{v}}{\partial t} + \frac{\partial \vec{u}}{\partial t} \times \vec{v} \tag{3.10}$$

$$\frac{\partial}{\partial t}(\phi \vec{u}) = \phi \frac{\partial \vec{u}}{\partial t} + \frac{\partial \phi}{\partial t} \vec{u} \tag{3.11}$$

$$\frac{\partial}{\partial t}(\vec{u} \cdot \vec{v} \times \vec{w}) = \vec{u} \cdot \vec{v} \times \frac{\partial \vec{w}}{\partial t} + \vec{u} \cdot \frac{\partial \vec{v}}{\partial t} \times \vec{w} + \frac{\partial \vec{u}}{\partial t} \cdot \vec{v} \times \vec{w} \tag{3.12}$$

$$\frac{\partial}{\partial t}\{\vec{u} \times (\vec{v} \times \vec{w})\} = \vec{u} \times \left(\vec{v} \times \frac{\partial \vec{w}}{\partial t}\right) + \vec{u} \times \left(\frac{\partial \vec{v}}{\partial t} \times \vec{w}\right) + \frac{\partial \vec{u}}{\partial t} \times (\vec{v} \times \vec{w}) \tag{3.13}$$

Proposition 3.3

Let $\vec{u}$ and $\vec{v}$ are vector function of x, y, z. Then the following laws hold:

$$\text{If } \vec{u} = u_x \hat{i} + u_y \hat{j} + u_z \hat{k}, \text{ then } d\vec{u} = du_x \hat{i} + du_y \hat{j} + du_z \hat{k} \tag{3.14}$$

$$d(\vec{u} \cdot \vec{v}) = \vec{u} \cdot d\vec{v} + d\vec{u} \cdot \vec{v} \tag{3.15}$$

$$d(\vec{u} \times \vec{v}) = \vec{u} \times d\vec{v} + d\vec{u} \times \vec{v} \tag{3.16}$$

3.2 Point Function

Definition 3.1 (Point function)

A physical quantity can be expressed as a continuous function of the position of the point in a region of space, such a function is called point function *and the region in which it specifies the physical quantity is known as* field.

Definition 3.2 (Scalar point function)

A scalar quantity can be expressed as a continuous function of the position of the point in a region of space, such a function is called scalar point function *and the region in which it specifies the physical quantity is known as a* scalar field.

If $P(x, y, z)$ is a point in the region, then $\phi(x, y, z)$ defines a scalar point function or a scalar field for the region.

Example 3.1 $\phi(x, y, z) = 3x^2 - z^3x - zy$ defines a scalar field. The temperature at any point within or on the surface at a certain time defines a scalar field.

Example 3.2 Temperature, density etc. scalar quantities can be expressed by scalar point functions.

- The temperature distribution within some body at a particular point in time.
- The density distribution within some fluid at a particular point in time.

Definition 3.3 (Vector point function)
A vector quantity can be expressed as a continuous function of the position of the point in a region of space, such a function is called vector point function *and the region in which it specifies the physical quantity is known as a* vector field. ♣

If $P(x, y, z)$ is a point in the region, then $\vec{v}(x, y, z)$ defines a vector point function or a vector field for the region.

Example 3.3 If the velocity at a point $P(x, y, z)$, which in a moving fluid is known for the region, then $\vec{v}$(P) is the vector field.

$$\vec{v}(x, y, z) = xy\hat{i} + zx\hat{j} + yz\hat{k},$$

is a example of vector field.

Example 3.4 Following vector quantities can be expressed by vector point functions.

- Gravitational field of the earth.
- Electric field about a current-carrying wire.
- Magnetic field generated by a magnet.
- Velocity at different points within a moving fluid.
- Acceleration at different points within a moving fluid.

3.3 Differentiation of Vectors

The vector differential operator del, written ∇, is defined as follows:

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \tag{3.17}$$

This vector operator possesses properties analogous to those of ordinary vectors. It is useful in defining three quantities that appear in applications and which are known as the gradient, the divergence, and the curl. The operator ∇ is also known as nabla.

3.4 Gradient

Definition 3.4 (Gradient)
Let $\phi(x, y, z)$ be a scalar function defined and differentiable at each point (x, y, z) in a certain region of space. [That is, ϕ defines a differentiable scalar field.] Then the gradient of ϕ *, written $\nabla\phi$ or grad ϕ is defined as follows:*

$$\nabla\phi = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} = \sum\frac{\partial\phi}{\partial x}\hat{i}.$$

♣

Note *$\nabla\phi$ defines a vector field.*

Problem 3.1 Let $\phi(x, y, z) = 3xy^2 - y^2z^2$. Find $\nabla\phi$ (or grad ϕ) at the point $P(1, 1, 2)$.

Solution

$$\begin{aligned}\nabla\phi &= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(3xy^2-y^2z^2\right)\\ &= 3y^2\hat{\imath}+\left(6xy-2yz^2\right)\hat{\jmath}-2y^2z\hat{k}\end{aligned}$$

Therefore,

$$\nabla\phi(1,1,2)=3\hat{\imath}+(6-8)\,\hat{\jmath}-4\hat{k}=3\hat{\imath}-2\hat{\jmath}-4\hat{k}.$$

3.4.1 Directional Derivative of a Scalar Point Function

We can find directional derivative of a scalar point function along any line or a vector by following definition.

Definition 3.5 (Directional Derivative)

Consider a scalar function $\phi=\phi(x,y,z)$. Then the directional derivative of ϕ in the direction of a vector $\vec{u}$ *is denoted by $D_{\vec{u}}(\phi)=\nabla\phi\cdot\hat{u}$. Where $\hat{u}$, is the unite vector of $\vec{u}$.* ♣

Problem 3.2 Consider the scalar function $\phi(x,y,z)=x^2+y^2+xz$.

1. Find grad ϕ.
2. Find grad ϕ at the point $P=P(2,-1,3)$.
3. Find the direction derivative of ϕ at the point P in the direction of $\vec{u}=\hat{\imath}+2\hat{\jmath}+\hat{k}$.

Solution

$$\begin{aligned}\nabla\phi &= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(x^2+y^2+xz\right)\\ &= (2x+z)\,\hat{\imath}+2y\hat{\jmath}+x\hat{k}\end{aligned}$$

Therefore,

$$\nabla\phi(2,-1,3)=7\hat{\imath}-2\hat{\jmath}+2\hat{k}.$$

Now, for unite vector $\hat{u}$ in the direction of $\vec{u}$,

$$\hat{u}=\frac{\vec{u}}{u}=\frac{2\hat{\imath}-\hat{\jmath}+3\hat{k}}{\sqrt{2^2+1^2+3^2}}=\frac{1}{\sqrt{14}}\left(2\hat{\imath}-\hat{\jmath}+3\hat{k}\right)$$

Then the directional derivative of ϕ at the point $P(2,-1,3)$ in the direction of $\vec{u}$ follows:

$$\nabla\phi\cdot\hat{u}=\left(7\hat{\imath}-2\hat{\jmath}+2\hat{k}\right)\cdot\left[\frac{1}{\sqrt{14}}\left(2\hat{\imath}-\hat{\jmath}+3\hat{k}\right)\right]=\frac{14+2+6}{\sqrt{14}}=\frac{22}{\sqrt{14}}.$$

3.5 Divergence

Definition 3.6 (Divergence)

Let $\vec{u}(x,y,z)=u_1\hat{\imath}+u_2\hat{\jmath}+u_3\hat{k}$ is defined and differentiable at each point (x,y,z) in a region of space. (That is, $\vec{u}$ defines a differentiable vector field.) Then the divergence *of $\vec{u}$, written $\nabla\cdot\vec{u}$*

or div $\vec{u}$ is defined as follows:

$$\begin{aligned} \text{div}\,\vec{u} = \nabla\cdot\vec{u} &= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(u_1\hat{i} + u_2\hat{j} + u_3\hat{k}\right), \\ &= \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}. \\ &= \sum \frac{\partial u_1}{\partial x}. \end{aligned}$$

♣

Note

1. *Although $\vec{u}$ is a vector, $\nabla\cdot\vec{u}$ is a scalar.*
2. $\nabla\cdot\vec{u} \neq \vec{u}\cdot\nabla$.
3. $\nabla\cdot\vec{u}$ *is a scalar.*
4. $\vec{u}\cdot\nabla$ *is a operator.*

Problem 3.3 Let $\vec{u} = x^2z^2\hat{i} - 2y^2z^2\hat{j} + xy^2z\hat{k}$. Find $\nabla\cdot\vec{u}$ at the point $P(1,-1,1)$.

Solution

$$\begin{aligned} \nabla\cdot\vec{u} &= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(x^2z^2\hat{i} - 2y^2z^2\hat{j} + xy^2z\hat{k}\right), \\ &= \frac{\partial}{\partial x}\left(x^2z^2\right) + \frac{\partial}{\partial y}\left(-2y^2z^2\right) + \frac{\partial}{\partial z}\left(xy^2z\right) \\ &= 2xz^2 - 4yz^2 + xy^2 \end{aligned}$$

At the point $P(1,-1,1)$,

$$\nabla\cdot\vec{u} = 2 + 4 + 1 = 7.$$

Definition 3.7 (Solenoidal)

A vector $\vec{u}$ is said to be solenoidal *if* $\nabla\cdot\vec{u} = 0$.

♣

3.6 Curl

Definition 3.8 (Curl)

Let $\vec{u}(x,y,z) = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ is defined and differentiable at each point (x,y,z) in a region of space. (That is, $\vec{u}$ defines a differentiable vector field.) Then the curl *of $\vec{u}$, written $\nabla\times\vec{u}$ or* *curl $\vec{u}$ is defined as follows:*

$$\begin{aligned}
curl\ \vec{u} = \nabla \times \vec{u} &= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times \left(u_1\hat{i} + u_2\hat{j} + u_3\hat{k}\right), \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_1 & u_2 & u_3 \end{vmatrix} \\
&= \left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right)\hat{i} + \left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}\right)\hat{j} + \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right)\hat{k} \\
&= \sum \left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right)\hat{i} \qquad (3.18)
\end{aligned}$$

♣

Definition 3.9 (Irrotational)

A vector $\vec{u}$ is said to be irrotational *if* $\nabla \times \vec{u} = 0$.

♣

3.7 Formulae involving ∇

Proposition 3.4

If ϕ, and ψ are differentiable scalar function of positions (x, y, z) then prove that grad $(\phi + \psi) =$ grad $\phi +$ grad ψ, or

$$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi.$$

♠

Proof

$$\begin{aligned}
\nabla(\phi + \psi) &= \frac{\partial(\phi + \psi)}{\partial x}\hat{i} + \frac{\partial(\phi + \psi)}{\partial y}\hat{j} + \frac{\partial(\phi + \psi)}{\partial z}\hat{k} \\
&= \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right) + \left(\frac{\partial\psi}{\partial x}\hat{i} + \frac{\partial\psi}{\partial y}\hat{j} + \frac{\partial\psi}{\partial z}\hat{k}\right) \\
&= \nabla\phi + \nabla\psi.
\end{aligned}$$

Proposition 3.5

If $\vec{u}$, and $\vec{v}$ are differentiable vector function of positions (x, y, z) then prove that div $(\phi + \psi) =$ div $\phi +$ div ψ, or

$$\nabla \cdot (\vec{u} + \vec{v}) = \nabla \cdot \vec{u} + \nabla \cdot \vec{v}.$$

♠

Proof

$$\begin{aligned}
\nabla \cdot (\vec{u} + \vec{v}) &= \sum \hat{i}\frac{\partial}{\partial x} \cdot (\vec{u} + \vec{v}), \\
&= \sum \hat{i} \cdot \left(\frac{\partial}{\partial x}\vec{u} + \frac{\partial}{\partial x}\vec{v}\right) \\
&= \sum \hat{i} \cdot \frac{\partial}{\partial x}\vec{u} + \sum \hat{i} \cdot \frac{\partial}{\partial x}\vec{v} \\
&= \nabla \cdot \vec{u} + \nabla \cdot \vec{v}.
\end{aligned}$$

Proposition 3.6

If $\vec{u}$, and $\vec{v}$ are differentiable vector function of positions (x, y, z) then prove that curl $(\phi + \psi) =$ curl $\phi +$ curl ψ, or

$$\nabla \times (\vec{u} + \vec{v}) = \nabla \times \vec{u} + \nabla \times \vec{v}.$$

Proof

$$\begin{aligned}
\nabla \times (\vec{u} + \vec{v}) &= \sum \hat{i} \frac{\partial}{\partial x} \times (\vec{u} + \vec{v}), \\
&= \sum \hat{i} \times \left(\frac{\partial}{\partial x} \vec{u} + \frac{\partial}{\partial x} \vec{v} \right) \\
&= \sum \hat{i} \times \frac{\partial}{\partial x} \vec{u} + \sum \hat{i} \times \frac{\partial}{\partial x} \vec{v} \\
&= \nabla \times \vec{u} + \nabla \cdot \vec{v}.
\end{aligned}$$

Proposition 3.7

If ϕ, and ψ are differentiable scalar function of positions (x, y, z) then prove that grad $(\phi\psi) =$ ϕ grad $\psi + \psi$ grad ψ, or

$$\nabla (\phi\psi) = \phi \nabla \psi + \psi \nabla \phi.$$

Proof

$$\begin{aligned}
\nabla (\phi\psi) &= \sum \hat{i} \frac{\partial}{\partial x} (\phi\psi), \\
&= \sum \hat{i} \left(\phi \frac{\partial}{\partial x} \psi + \psi \frac{\partial}{\partial x} \phi \right) \\
&= \phi \sum \hat{i} \frac{\partial}{\partial x} \psi + \psi \sum \hat{i} \frac{\partial}{\partial x} \phi \\
&= \phi \nabla \psi + \psi \nabla \phi.
\end{aligned}$$

Proposition 3.8

If $\vec{u}$, and $\vec{v}$ are differentiable vector function of positions (x, y, z) then prove that grad $(\vec{u} \cdot \vec{v}) =$ $\vec{u} \times$ curl $\vec{v} + \vec{v} \times$ curl $\vec{u} + (\vec{u} \cdot \nabla) \vec{v} + (\vec{v} \cdot \nabla) \vec{u}$, or

$$\nabla (\vec{u} \cdot \vec{v}) = \vec{u} \times \nabla \times \vec{v} + \vec{v} \times \nabla \times \vec{u} + (\vec{u} \cdot \nabla) \vec{v} + (\vec{v} \cdot \nabla) \vec{u}.$$

Proof

$$\begin{aligned}
\nabla (\vec{u} \cdot \vec{v}) &= \sum \hat{i} \frac{\partial}{\partial x} (\vec{u} \cdot \vec{v}), \\
&= \sum \hat{i} \left(\vec{u} \cdot \frac{\partial}{\partial x} \vec{v} + \vec{v} \cdot \frac{\partial}{\partial x} \vec{u} \right) \\
&= \sum \hat{i} \left(\vec{u} \cdot \frac{\partial}{\partial x} \vec{v} \right) + \sum \hat{i} \left(\vec{v} \cdot \frac{\partial}{\partial x} \vec{u} \right) \qquad (3.19)
\end{aligned}$$

Now,

$$\begin{aligned}
\vec{u}\times\nabla\times\vec{v} &= \vec{u}\times\sum\hat{i}\frac{\partial}{\partial x}\times\vec{v}\\
&= \vec{u}\times\sum\hat{i}\times\frac{\partial\vec{v}}{\partial x}\\
&= \sum\hat{i}\left(\vec{u}\cdot\frac{\partial}{\partial x}\vec{v}\right)-\sum\left(\vec{u}\cdot\hat{i}\right)\frac{\partial\vec{v}}{\partial x}\\
&= \sum\hat{i}\left(\vec{u}\cdot\frac{\partial}{\partial x}\vec{v}\right)-\sum\left(\vec{u}\cdot\hat{i}\frac{\partial}{\partial x}\right)\vec{v}\\
&= \sum\hat{i}\left(\vec{u}\cdot\frac{\partial}{\partial x}\vec{v}\right)-\sum\left(\vec{u}\cdot\nabla\right)\vec{v}\\
\implies\sum\hat{i}\left(\vec{u}\cdot\frac{\partial}{\partial x}\vec{v}\right) &= \vec{u}\times\nabla\times\vec{v}+\sum\left(\vec{u}\cdot\nabla\right)\vec{v}
\end{aligned}$$

Similarly,

$$\sum\hat{i}\left(\vec{v}\cdot\frac{\partial}{\partial x}\vec{u}\right) = \vec{v}\times\nabla\times\vec{u}+\sum\left(\vec{v}\cdot\nabla\right)\vec{u}$$

Chapter 3 Exercise

1. Define following
   (a). Point function
   (b). Scalar point function
   (c). Vector point function
   (d). Gradient
   (e). Divergence
   (f). Curl
   (g). Solenoidal
   (h). Irrotational
2. If $\vec{u}$, and $\vec{v}$ are differentiable vector function and ϕ, and ψ are differentiable scalar function of positions (x, y, z) then prove the following formulae
   (a). $\nabla(\phi+\psi)=\nabla\phi+\nabla\psi$,
   (b). $\nabla\cdot(\vec{u}+\vec{v})=\nabla\cdot\vec{u}+\nabla\cdot\vec{v}$,
   (c). $\nabla\times(\vec{u}+\vec{v})=\nabla\times\vec{u}+\nabla\times\vec{v}$,
   (d). $\nabla\cdot(\phi\vec{u})=(\nabla\phi)\cdot\vec{u}+\phi(\nabla\cdot\vec{u})$,
   (e). $\nabla\times(\phi\vec{u})=(\nabla\phi)\times\vec{u}+\phi(\nabla\times\vec{u})$,
   (f). $\nabla\cdot(\vec{u}\times\vec{v})=\vec{u}\cdot(\nabla\vec{v})-\vec{u}\cdot(\nabla\times\vec{v})$,
   (g). $\nabla\times(\vec{u}\times\vec{v})=(\vec{v}\cdot\nabla)\vec{u}-\vec{v}(\nabla\cdot\vec{u})-(\vec{u}\cdot\nabla)\vec{v}+\vec{u}(\nabla\cdot\vec{v})$,
   (h). $\nabla(\vec{u}\cdot\vec{v})=(\vec{v}\cdot\nabla)\vec{u}+(\vec{u}\cdot\nabla)\vec{v}+\vec{v}\times(\nabla\times\vec{u})+\vec{u}\times(\nabla\times\vec{v})$,
   (i). $\nabla\cdot(\nabla\phi)=\nabla^2\phi=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi$,
   (j). $\nabla\times(\nabla\phi)=$ curl grad $\phi=0$,
   (k). $\nabla\cdot(\nabla\times\vec{u})=$ div curl $\vec{u}=0$,
   (l). $\nabla\times(\nabla\times\vec{u})=\nabla(\nabla\cdot\vec{u})-\nabla^2\vec{u}$

3. Let $\vec{r} = (x, y, z)$, then prove the following formulae
   (a). div $\vec{r} = \nabla \cdot \vec{r} = 3$,
   (b). curl $\vec{r} = \nabla \times \vec{r} = 0$,
   (c). $\nabla \cdot \left(\frac{\vec{r}}{r^3}\right) = \text{ div } \left(\frac{\vec{r}}{r^3}\right) = 0$,
   (d). $\nabla \cdot (r^n \vec{r}) = \text{ div } (r^n \vec{r}) = (n+3) r^n$

Chapter 4 Integration of Vectors

Introduction

- *Vector Integration*
- *Line Integral*
- *Surface Integral*
- *Volume Integral*
- *Green's Theorem*
- *Gauss's Theorem*
- *Stokes's Theorem*
- *Exercise*

4.1 Vector Integration

Definition 4.1 (Vector Integration)

Let $\vec{V} = \vec{V}(t) = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$ be a given vector function of the scalar variable t, and these values are supposed to be finite and for values of t in specified interval. Then

$$\int V(t)\,dt = \hat{i}\int V(t)\,dt + \hat{j}\int V(t)\,dt + \hat{k}\int V(t)\,dt \tag{4.1}$$

is called the indefinite integral *of $\vec{V}(t)$.* ♣

4.2 Line Integral

Definition 4.2 (Line Integral)

The line integral *of $\vec{F}$ along the curve C may be written as*

$$\int_C \vec{F}\cdot d\vec{r}. \tag{4.2}$$

♣

Problem 4.1 Find the line integral $\oint y^2dx - x^2dy$ about the triangle whose vertices are (1,0), (0,1), (-1,0).

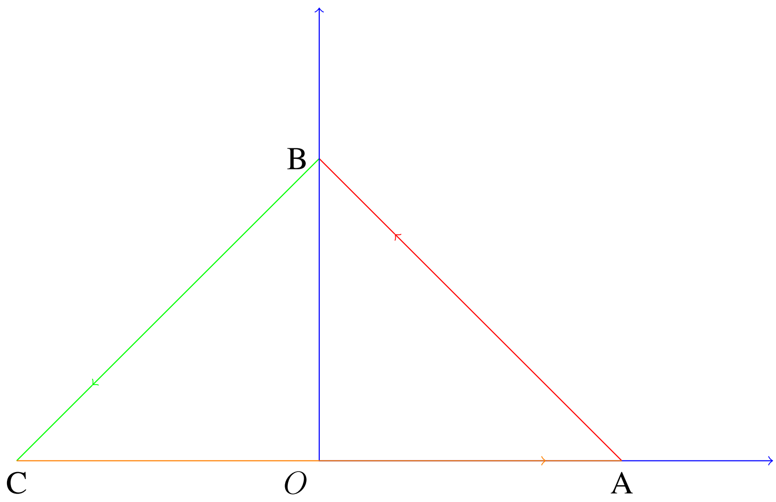
Solution *The sides of the triangles are the segments of the lines*

$$\frac{x-1}{1-0} = \frac{y-0}{0-1}; \quad \frac{x-0}{0+1} = \frac{y-1}{1-0}; \quad \frac{x+1}{-1-1} = \frac{y-0}{0-0};$$

or,

$$x + y = 1; \quad ,y - x = 1; \quad y = 0. \tag{4.3}$$

Hence, line integrals are



$$
\begin{aligned}
I_1 &= \int_{C_1} (y^2 dx - x^2 dy) \\
&= \int_1^0 ((1-x)^2 dx + x^2 dx) \quad [\textit{for } y = 1 - x; dy = -dx] \\
&= \left[-\frac{(1-x)^3}{3} + \frac{x^3}{3}\right]_1^0 = \frac{-1}{3} + 0 - 0 - \frac{1}{3} = -2/3. \\
I_2 &= \int_{C_2} (y^2 dx - x^2 dy) \\
&= \int_0^{-1} ((x+1)^2 dx - x^2 dx) \quad [\textit{for } y = 1 + x;\ dy = dx] \\
&= \left[\frac{(x+1)^3}{3} - \frac{x^3}{3}\right]_0^{-1} = -\frac{1}{3} + \frac{1}{3} = 0,
\end{aligned}
$$

and

$$
I_3 = \int_{C_3} (y^2 dx - x^2 dy) = 0 \quad [\textit{for } y = 0; dy = 0]
$$

then $I = I_1 + I_2 + I_3 = -\frac{2}{3}$.

Problem 4.2 Find the line integral $\oint y^2 dx - x^2 dy$ about the circle $x^2 + (y-1)^2 = 1$.

Solution *If* $x = \cos\theta$*, then* $y = 1 + \sin\theta$*, which represents the circle. Also* $dx = -\sin\theta$*, and* $dy = \cos\theta$*. Then*

$$
\begin{aligned}
I &= \oint_C (y^2 dx - x^2 dy) \\
&= \int_0^{2\pi} \left[-(1+\sin\theta)^2 \sin\theta - \cos^2\theta\cos\theta\right] d\theta \\
&= -\int_0^{2\pi} \left(\sin\theta + 2\sin^2\theta + \sin^3\theta + \cos^3\theta\right) d\theta \\
&= -\int_0^{2\pi} \left(\sin\theta + (1 - \cos 2\theta) + \frac{1}{4}(3\sin\theta - \sin 3\theta) + \frac{1}{4}(3\cos\theta + \cos 3\theta)\right) d\theta \\
&= -[\theta]_0^{2\pi} = -2\pi. \qquad \left[\because \int_0^{2\pi} \sin\theta = 0; \quad \int_0^{2\pi} \cos\theta = 0.\right]
\end{aligned}
$$

4.3 Surface Integral

Definition 4.3

The surface integral *of the vector $\vec{F} = \vec{F}(r)$ over the surfaces S is defined to be*

$$\int_S \vec{F} \cdot d\vec{a}. \tag{4.4}$$

♣

Definition 4.4 (Simple Closed Curve)

A closed curve which does not intersects itself is called a simple closed curve. ♣

Definition 4.5 (Simply Connected Region)

If a plane region has the property that any simple closed curve in it can can be continuously shrunk to a point without leaving the region then the region is called a simply connected region. ♣

Definition 4.6 (Multiply Connected Region)

If a plane region has the property that any simple closed curve in it can can not be continuously shrunk to a point without leaving the region then the region is called a multiply connected region. ♣

4.4 Volume Integral

Definition 4.7 (Volume Integral)

If we consider a closed surface in space enclosing a volume V, then

$$V = \iiint \vec{F} \cdot d\vec{v}, \tag{4.5}$$

is defined the volume integration. ♣

4.5 Green's Theorem

Definition 4.8 (Green's Theorem)

Let R be a closed region bounded by a closed regular curve C whose boundary is cut at most two points by parallel to axes. Then if $M(x, y)$, $N(x, y)$, $\frac{\partial M}{\partial y}$, $\frac{\partial N}{\partial x}$ are continuous in R then

$$\oint_C (Mdx + Ndx) = \oint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy \tag{4.6}$$

where the circuit integral C is taken over the boundary of R in the positive sense. ♣

Problem 4.3 Verify Green's Theorem in the plane for $\oint_C \{(2x - y^3)\, dx - xydy\}$, where C is the boundary of the region enclosed by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$.

Solution *Along the circle $x^2+y^2=1$, the integral is*

$$\oint_{C_1}\left\{\left(2x-y^3\right)dx-xydy\right\}$$

Let $x=\cos\theta$, and $y=\sin\theta$ then $dx=-\sin\theta d\theta$, and $dy=\cos\theta d\theta$ then we have,

$$\begin{aligned}\oint_{C_1}\left\{\left(2x-y^3\right)dx-xydy\right\} &= -\int_0^{2\pi}\left(2\cos\theta-\sin^3\theta\right)\left(-\sin\theta d\theta\right)-\cos\theta\sin\theta\cos\theta d\theta \\ &= -\int_0^{2\pi}\left(-2\cos\theta+\sin^3\theta-\cos^2\theta\right)\sin\theta d\theta \\ &= -\int_0^{2\pi}\sin^4\theta d\theta=-4\cdot\frac{3}{4}\frac{1}{2}\frac{\pi}{2}=-\frac{3\pi}{4}.\end{aligned}$$

Similarly, along the circle $x^2+y^2=9$, the integral is

$$\oint_{C_2}\left\{\left(2x-y^3\right)dx-xydy\right\}$$

Let $x=3\cos\theta$, and $y=3\sin\theta$ then $dx=-3\sin\theta d\theta$, and $dy=3\cos\theta d\theta$ then we have,

$$\begin{aligned}\oint_{C_2}\left\{\left(2x-y^3\right)dx-xydy\right\} &= \int_0^{2\pi}\left(6\cos\theta-27\sin^3\theta\right)\left(-3\sin\theta d\theta\right)-3\cos\theta 3\sin\theta 3\cos\theta d\theta \\ &= \int_0^{2\pi}\left(-18\cos\theta+81\sin^3\theta-27\cos^2\theta\right)\sin\theta d\theta \\ &= 81\int_0^{2\pi}\sin^4\theta d\theta=4\cdot 81\cdot\frac{3}{4}\frac{1}{2}\frac{\pi}{2}=\frac{243\pi}{4}.\end{aligned}$$

Hence,

$$\oint_C Mdx+Ndy=\frac{243\pi}{4}-\frac{3\pi}{4}=60\pi.$$

Again

$$I=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy=\iint_R\left(\frac{\partial}{\partial x}\left(-xy\right)-\frac{\partial}{\partial y}\left(2x-y^3\right)\right)dxdy=\iint_R\left(-x+3y^2\right)dxdy$$

Putting $x=r\cos\theta$, $y=r\sin\theta$, and $dxdy=rd\theta dr$. Limits of r from 0 to 1 and θ from 0 to 2π.

$$\begin{aligned}I &= \int_1^3\int_0^{2\pi}\left(-r\cos\theta+3r^2\sin^2\theta\right)rd\theta dr=\int_1^3\left[-r\sin\theta+3r^2\left(\frac{1}{2}\theta-\frac{1}{4}\sin^2\theta\right)\right]_0^{2\pi}rdr \\ &= \int_1^3 3r^3\pi dr=\frac{3\pi}{4}(81-1)=60\pi.\end{aligned}$$

Hence, the theorem is verified.

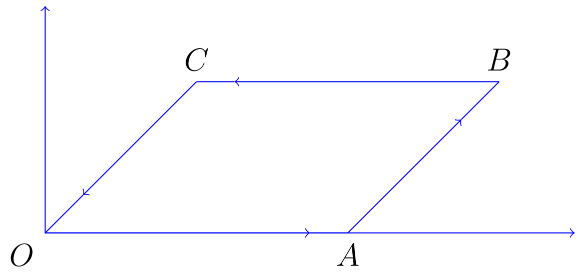
Problem 4.4 Verify Green's Theorem for $\oint_C\left(3x^2+2y\right)dx-(x+3\cos y)dy$, around the parallelogram having vertices at (0,0), (2,0), (3,1), and (1,1).

Solution *Along the line OA, $y=0$, and $dy=0$, hence, the integral is*

$$\oint_{C_1}\left(3x^2+2y\right)dx-(x+3\cos y)dy=\int_0^2 3x^2dx=\left[x^3\right]_0^2=8.$$

Along the line AB, $y = x - 2$, and $dy = dx$ then the integral is

$$\begin{aligned}\oint_{C_2} \left(3x^2 + 2y\right) dx - (x + 3\cos y)dy &= \int_2^3 \left(3x^2 + x - 4 - 3\cos(x-2)\right) dx \\ &= \left[x^3 + \frac{x^2}{2} - 4x - 3\sin(x-2)\right]_2^3 \\ &= 27 + \frac{9}{2} - 12 - 3\sin 1 - 8 - 2 + 8 - 0 = \frac{35}{2} - 3\sin 1.\end{aligned}$$



Along the line BC, $y = 1$, and $dy = 0$, hence, the integral is

$$\oint_{C_3} \left(3x^2 + 2y\right) dx - (x + 3\cos y)dy = \int_3^1 (3x^2 + 2)dx = \left[x^3 + 2x\right]_3^1 = 1 + 2 - 27 - 6 = -30.$$

And along the line CO, $y = x$, and $dy = dx$ then the integral is

$$\begin{aligned}\oint_{C_4} \left(3x^2 + 2y\right) dx - (x + 3\cos y)dy &= \int_1^0 \left(3x^2 + x - \cos x\right) dx \\ &= \left[x^3 + \frac{x^2}{2} - 3\sin x\right]_1^0 \\ &= -1 - \frac{1}{2} + 3\sin 1 = -\frac{3}{2} + 3\sin 1.\end{aligned}$$

Hence,

$$\oint_C \left(3x^2 + 2y\right) dx - (x + 3\cos y)dy = 8 + \frac{35}{2} - 3\sin 1 - 30 - \frac{3}{2} + 3\sin 1 = -6.$$

Again

$$I = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_R \left(\frac{\partial}{\partial x}(-x - 3\cos y) - \frac{\partial}{\partial y}\left(3x^2 + 2y\right)\right) dxdy = -3\iint_R dxdy$$

Along the line OA

$$I_1 = -3\int_{x=0}^{2}\int_{y=0}^{0} dydx = 0.$$

Along the line AB

$$I_2 = -3\int_{x=2}^{3}\int_{y=0}^{1} dydx = -3\int_{x=2}^{3} [y]_0^1\, dx = -3\int_{x=2}^{3} dx = -3$$

Along the line BC

$$I_3 = \int_{x=3}^{1}\int_{y=1}^{1} (6x + 3\sin y)\, dydx = 0.$$

And along the line CO

$$I_4 = -3\int_{x=1}^{0}\int_{y=1}^{0} dydx = -3\int_{x=2}^{3}[y]_1^0\,dx = 3\int_{x=2}^{3} dx = -3.$$

Which provide us

$$I = I_1 + I_2 + I_3 + I_4 = 0 - 3 + 0 - 3 = -6.$$

Hence, the theorem is verified.

4.6 Divergence Theorem of Gauss

Definition 4.9 (Divergence Theorem of Gauss)

Let V is the volume bounded by a closed surface S and $\vec{F}$ is a vector function of position with continuous derivatives. Then divergence theorem of Gauss *is given by*

$$\iiint_V div\,\vec{F}dv = \iiint_V \nabla\cdot\vec{F}dv = \iint_S \vec{F}\cdot\hat{n}ds \tag{4.7}$$

where $\hat{n}$ is the positive (outward drawn) normal to S. ♣

Problem 4.5 Verify the divergence theorem for the vector field, $\vec{F} = (2xy+z)\,\hat{i} + y^3\hat{j} - (x+3y)\,\hat{k}$ taken over the region bounded by $2x+2y+z=6$, $x=0$, $y=0$, $z=0$.

Solution

$$\begin{aligned}
\oint_V \nabla\vec{F}dv &= \oint_V\left[\frac{\partial}{\partial x}(2xy+z) + \frac{\partial}{\partial y}y^3 + \frac{\partial}{\partial z}(-x-3y)\right]dv \\
&= \oint_V \left[2y+3y^2\right]dv \\
&= \int_{x=0}^{3}\int_{y=0}^{3-x}\int_{z=0}^{6-2x-2y}\left[2y+3y^2\right]dzdydx \\
&= \int_{x=0}^{3}\int_{y=0}^{3-x}\left(2y+3y^2\right)[z]_0^{6-2x-2y}\,dydx \\
&= 2\int_{x=0}^{3}\int_{y=0}^{3-x}\left(2y+3y^2\right)(3-x-y)\,dydx \\
&= 2\int_{x=0}^{3}\int_{y=0}^{3-x}\left(6y+9y^2-2xy-3xy^2-2y^2-3y^3\right)dydx \\
&= 2\int_{x=0}^{3}\int_{y=0}^{3-x}\left(6y+7y^2-2xy-3xy^2-3y^3\right)dydx \\
&= 2\int_{x=0}^{3}\left[3y^2+\frac{7}{3}y^3-xy^2-xy^3-\frac{3}{4}y^4\right]_0^{3-x}dx \\
&= 2\int_{x=0}^{3}\left[(3-x)^3+\frac{7}{3}(3-x)^3-x(3-x)^3-\frac{3}{4}(3-x)^4\right]dx \\
&= 2\int_{x=0}^{3}(3-x)^3\left[\frac{10}{3}-x-\frac{9}{4}+\frac{3}{4}x\right]dx
\end{aligned}$$

$$
\begin{aligned}
&= 2\int_{x=0}^{3}(3-x)^3\left[\frac{13}{12}-\frac{1}{4}x\right]dx \\
&= \frac{1}{2}\int_{x=0}^{3}(3-x)^3\left[\frac{13}{3}-x\right]dx \\
&= \frac{1}{2}\int_{t=0}^{3}t^3\left[\frac{4}{3}+t\right]dt \qquad \left[\text{let } \begin{array}{c} t=3-x \implies x=3-t \implies dx=-dt \\ x=0,t=3;x=3,t=0 \end{array}\right] \\
&= \frac{1}{2}\int_{t=0}^{3}\left[\frac{4}{3}t^3+t^4\right]dt \\
&= \frac{1}{2}\left[\frac{1}{3}t^4+\frac{1}{5}t^5\right]_0^3 = \frac{1}{2}\left[1+\frac{9}{5}\right]3^3 = \frac{1}{2}\frac{14}{5}27 = \frac{189}{5} \qquad (4.8)
\end{aligned}
$$

The surface S consists of four faces are $S_1(x=0)$, $S_2(y=0)$, $S_3(z=0)$, $S_4(2x+2y+z=6)$.

On the plane $x=0$, $nds=-\hat{i}dxdz$.

$$\oint_{S_1}\vec{F}\cdot nds = \oint_R\left(z\hat{i}+y^3\hat{j}-3y\hat{k}\right)(-\hat{i})dxdz = -\int_{x=0}^{3}\int_{y=0}^{\frac{1}{2}(6-z)}zdydz = 0.$$

On the plane $y=0$, $nds=-\hat{j}dxdz$.

$$\oint_{S_2}\vec{F}\cdot nds = \oint_R\left(z\hat{i}-x\hat{k}\right)(-\hat{j})dxdz = 0.$$

On the plane $z=0$, $nds=-\hat{k}dxdz$.

$$
\begin{aligned}
\oint_{S_3}\vec{F}\cdot nds &= \oint_R\left(2xy\hat{i}+y^3\hat{j}-(x+3y)\hat{k}\right)(-\hat{k})dxdy \\
&= \int_{y=0}^{3}\int_{x=0}^{3-y}(x+3y)dxdy \\
&= \int_{y=0}^{3}\left[\frac{x^2}{2}+3xy\right]_{x=0}^{3-y}dy \\
&= \int_{y=0}^{3}\frac{(3-y)^2}{2}+3y(3-y)dy \\
&= \frac{1}{2}\int_{y=0}^{3}(9+12y-5y^2)dy \\
&= \frac{1}{2}\left[9y+6y^2-\frac{5}{3}y^3\right]_{y=0}^{3} \\
&= \frac{1}{2}[27+54-45] = 18.
\end{aligned}
$$

On the plane

$$2x+2y+z=6, \implies \nabla(2x+2y+z) = 2\hat{i}+2\hat{j}+\hat{k},$$

which provide us $\hat{n} = \frac{2\hat{i}+2\hat{j}+\hat{k}}{\sqrt{2^2+2^2+1}} = \frac{2\hat{i}+2\hat{j}+\hat{k}}{3}$; $ds = 3dxdy$

$$\begin{aligned}
\oint_{S_4} \vec{F}\cdot nds &= \oint_R \left(2xy\hat{i} + y^3\hat{j} - (x+3y)\hat{k}\right)\cdot\frac{1}{3}\left(2\hat{i}+2\hat{j}+\hat{k}\right)3dxdy \\
&= \int\limits_{y=0}^{3}\int\limits_{x=0}^{3-y}\left(4xy + 2y^3 - x - 3y\right)dxdy \\
&= \int\limits_{y=0}^{3}\int\limits_{x=0}^{3-y}\left(x\left(4y-1\right) + 2y^3 - 3y\right)dxdy \\
&= \int\limits_{y=0}^{3}\left[\frac{1}{2}x^2\left(4y-1\right) + \left(2y^3-3y\right)x\right]_0^{3-y} dy \\
&= \int\limits_{y=0}^{3}\left[\frac{1}{2}\left(3-y\right)^2\left(4y-1\right) + \left(2y^3-3y\right)\left(3-y\right)\right] dy \\
&= \frac{1}{2}\int\limits_{y=0}^{3}\left[36y - 24y^2 + 4y^3 - 9 + 6y - y^2 + 12y^3 - 18y - 4y^4 + 6y^2\right] dy \\
&= \frac{1}{2}\int\limits_{y=0}^{3}\left[-9 + 24y - 19y^2 + 16y^3 - 4y^4\right] dy \\
&= \frac{1}{2}\left[-9y + 12y^2 - \frac{19}{3}y^3 + 4y^4 - \frac{4}{5}y^5\right]_0^3 \\
&= \frac{1}{2}\left[-27 + 108 - 171 + 324 - \frac{972}{5}\right] \\
&= \left[117 - \frac{486}{5}\right] = \frac{1}{3}\left[\frac{99}{5}\right]
\end{aligned}$$

Now we have,

$$\oint_S \vec{F}\cdot nds = \oint_{S_1}\vec{F}\cdot nds + \oint_{S_2}\vec{F}\cdot nds + \oint_{S_3}\vec{F}\cdot nds + \oint_{S_4}\vec{F}\cdot nds = 0 + 0 + 18 + \frac{99}{5} = \frac{189}{5}.$$

Hence, the theorem is verified.

4.7 Stokes's Theorem

Definition 4.10 (Stokes's Theorem)

Let C is the boundary of region enclosed by the surface S, the vector function $\vec{F}$ is single valued and continuous with first order derivative in any direction, then Stokes's theorem *is given by*

$$\int_C \vec{F}\cdot dl = \iint_S \hat{n}\ curl\ \vec{F}ds = \iint_S\left(\nabla\times\vec{F}ds\right) \tag{4.9}$$

♣

Problem 4.6 Verify the Stokes's theorem for the vector, $\vec{F} = 2y\hat{i} + 3x\hat{j} - z^2\hat{k}$, where S is the upper half surface of the sphere $x^2+y^2+z^2=9$ and C is boundary.

Solution *The boundary C of the region R is a circle in the xy-plane of radius 3 and center is at the origin. If we consider $x^2+y^2=9$, $z=0$, $x=3\cos\theta$, $y=3\sin\theta$, where $0<\theta<2\pi$. then*

$$\begin{aligned}
\oint_C \vec{F}\cdot dr &= \oint_C \left(2y\hat{i}+3x\hat{j}-z^2\hat{k}\right)\cdot\left(dx\hat{i}+dy\hat{j}+dz\hat{k}\right)\\
&= \oint_C \left(2ydx+3xdy-z^2dz\right)\\
&= \int_0^{2\pi}\left(-6\sin\theta 3\sin\theta d\theta+9\cos\theta 3\cos\theta d\theta\right)\\
&= 36\int_0^{\pi/2}\left(-2\sin^2\theta+3\cos^2\theta\right)d\theta\\
&= 36\int_0^{\pi/2}\left(\cos 2\theta-1+\frac{3}{2}\cos 2\theta+\frac{3}{2}\right)d\theta\\
&= 18\int_0^{\pi/2}\left(5\cos 2\theta+1\right)d\theta\\
&= 18\left[\frac{5}{2}\sin 2\theta+\theta\right]_0^{\pi/2}=9\pi.
\end{aligned}$$

Again

$$\nabla\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2y & 3x & -z^2\end{vmatrix}=(3-2)\hat{k}=\hat{k}.$$

Now

$$\begin{aligned}
\int_S\left(\nabla\times\vec{F}\right)\cdot\hat{n}ds &= \int_S\hat{k}\cdot\hat{n}ds=\int_R dxdy\\
&= \int_{x=-3}^{3}\int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}}dydx=4\int_{x=0}^{3}\int_{y=0}^{\sqrt{9-x^2}}dydx\\
&= 4\int_{x=0}^{3}[y]_0^{\sqrt{9-x^2}}dx=4\int_{x=0}^{3}\sqrt{9-x^2}dx\\
&= 4\left[\frac{x\sqrt{9-x^2}}{2}+\frac{9}{2}\sin^{-1}\frac{x}{3}\right]_0^3=4\frac{9}{2}\frac{\pi}{2}=9\pi.
\end{aligned}$$

Hence,

$$\oint_C\vec{F}\cdot dr=\int_S\left(\nabla\times\vec{F}\right)\cdot\hat{n}ds.$$

Stoke's theorem is verified.

Chapter 4 Exercise

1. Short questions.
   (a). State the line integral.
   (b). Define Surface integral.
   (c). Define Volume integral.
   (d). State Green's theorem in the plane.

(e). State the divergent theorem (Gauss's theorem).

(f). State the Stokes's theorem.

2. Find the line integral $\oint y^2dx - x^2dy$ about the triangle whose vertices are (1,0), (0,1), (-1,0).
3. Find the line integral $\oint y^2dx - x^2dy$ about the circle $x^2 + (y-1)^2 = 1$.
4. Verify the divergence theorem for the vector field, $\vec{F} = (2xy + z)\,\hat{i} + y^3\hat{j} - (x + 3y)\,\hat{k}$ taken over the region bounded by $2x + 2y + z = 6$, $x = 0$, $y = 0$, $z = 0$.
5. Verify Green's Theorem in the plane for $\oint_C \{(2x - y^3)\,dx - xydy\}$, where C is the boundary of the region enclosed by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$.
6. Verify Green's Theorem for $\oint_C (3x^2 + 2y)\,dx - (x + 3\cos(y))dy$, around the parallelogram having vertices at (0,0), (2,0), (3,1), and (1,1).
7. Verify the Stokes's theorem for the vector, $\vec{F} = 2y\hat{i} + 3x\hat{j} - z^2\hat{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 9$ and C is boundary.

Chapter 5 Complex Numbers

Introduction

- *Complex Number System*
- *Polar Form of Complex Number*
- *Exponential Form of Complex Number*
- *Operation in Polar Form*
- *De'Moiver's Theorem*
- *Roots of Complex Number*

5.1 Complex Number System

There is no number x that satisfies the polynomial equation $x^2 + 1 = 0$. To permit solution of this, and similar type equations, the set of complex number is introduced.

Definition 5.1 (Complex Number)

A complex number *can be written as $a + ib$, where $a, b \in \mathbb{R}$ where a, b are called real and imaginary parts respectively, and $i = \sqrt{-1}$, which is called the* imaginary unite.

We can consider real number as a subset of the set of complex number with $b = 0$. The complex number $0 + i0$ is corresponds to the real number 0.

If $z = a + ib$ be a complex number, then

- Real part of $z = \text{Re}(z) = a$.
- Imaginary part of $z = \text{Im}(z) = b$.

Two complex numbers $z_1 = a + ib$, and $z_2 = c + id$ are said to be equal if and only if $a = c$, and $b = d$.

Note *Inequalities for complex numbers are not defined.*

Definition 5.2 (Complex Conjugate)

The complex conjugate *of $z = a + ib$ is denoted by z^* or $\bar{z}$ and defined as $a - ib$.*

5.1.1 Operations of Complex Numbers

In performing operations with complex numbers, let $a, b, c, d, m \in \mathbb{R}$, then we can proceed as in the algebra of real numbers, replacing i^2 by -1 when it occurs.

1. Addition:

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

2. Multiplication:

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

,

$$m(a + ib) = ma + imb$$

3. Division: If c and d are not simultaneously zero, then
$$\frac{a+ib}{c+id} = \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}$$

5.2 Axiomatic Foundation of the complex Number System

From a strictly logical point of view, it is desirable to define a complex number as an ordered pair (a, b) of real numbers a and b subject to certain operational definitions, which turn out to be equivalent to those above. These definitions are as follows, where all letters represent real numbers.

1. Equality $(a, b) = (c, d)$, if and only if $a = c$, and $b = d$.
2. Sum $(a, b) + (c, d) = (a + c, b + d)$.
3. Multiplication
$$(a, b)(c, d) = (ac - bd, ad + bc)$$
$$m(a, b) = (ma, mb).$$

5.3 Polar and Exponential Form of Complex Numbers

Definition 5.3 (Absolute value or Modulus)

The absolute value *or* modulus *of a complex number $z = a + ib$ is defined as $|z| = |a + ib| = \sqrt{a^2 + b^2}$.* ♣

Definition 5.4 (Argument)

The argument *of a complex number $z = a + ib$ is defined as $\arg z = \theta = \tan^{-1}\left(\frac{b}{a}\right)$, it is a multivalued function.* ♣

Definition 5.5 (Argand Plane or Argand Diagram or Complex Plane)

When a complex number z is represented by a point $P(x, y)$ in the xy-plane, then this plane is called the argand plane/diagram *or* complex plane. ♣

5.3.1 Polar Form of Complex Number

Let P be a point in the complex plane corresponding to the complex number (a, b) or $a + ib$. Then we see from Fig. 5.1 that
$$a = r\cos\theta, \qquad b = r\sin\theta$$
where $r = \sqrt{a^2 + b^2} = |a+ib|$ is called the modulus or absolute value of $z = a+ib$ and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$, is called the amplitude or argument of z, is the angle that line OP makes with the positive x axis.

It follows that
$$z = a + ib = r\left(\cos\theta + i\sin\theta\right).$$

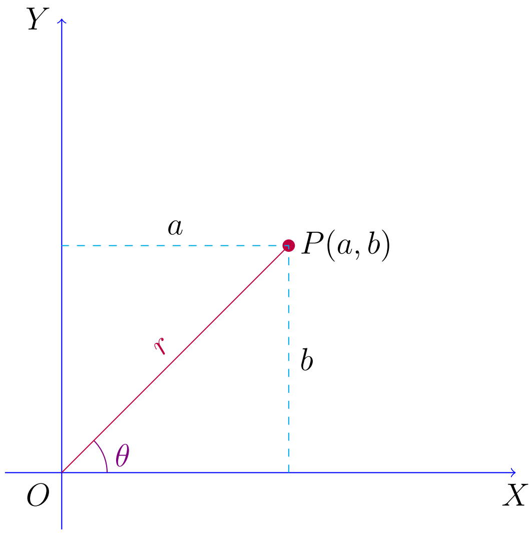


Figure 5.1: Polar form of a complex number.

which is called the polar form of the complex number, and r and θ are called polar coordinates.

5.3.2 Exponential Form of Complex Number

Definition 5.6 (Euler's formula)

Euler's formula *is given by,*

$$e^{i\theta} = \cos\theta + i\sin\theta$$

.

If $z = a + ib$ is a complex number, and (r, θ) is the polar form then

$$z = a + ib = r(\cos\theta + i\sin\theta) = re^{i\theta},$$

which is the exponential form of z.

Problem 5.1 Find the polar form of $-1 + i\sqrt{3}$.

Solution *Amplitude* $\theta = \tan^{-1}\left(-\sqrt{3}\right) = 2\pi/3$. *Modulus* $r = |-1 + i\sqrt{3}| = \sqrt{1+3} = 2$. *Then* $-1 + i\sqrt{3} = 2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$.

5.4 Operation in Polar Form

Proposition 5.1

If $z_1 = a_1 + ib_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, *and* $z_2 = a_2 + ib_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, *then*

$$z_1 z_2 = r_1 r_2 \{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\} = r_1 r_2 e^{i(\theta_1+\theta_2)} \tag{5.1}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} \tag{5.2}$$

Proposition 5.2

If $z_1, z_2 \in \mathbb{C}$, then following properties hold:

1. $|z_1 z_2| = |z_1||z_2|$
2. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

Proof Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$|z_1| = r_1, \text{ and } |z_2| = r_2.$$

Again $z_1 z_2 = r_1 r_2 \{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}$, which provide

$$|z_1 z_2| = r_1 r_2 \sqrt{\{\cos^2(\theta_1+\theta_2) + \sin^2(\theta_1+\theta_2)\}} = r_1 r_2 = |z_1||z_2|.$$

Similarly, can be proved $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$.

5.5 De Moivre's Theorem

Theorem 5.1 (De Moivre's Theorem)

De Moivre's theorem *state that*

$$(\cos\theta + i\sin\theta)^n = \cos^n\theta + i\sin^n\theta,$$

where n is any integer.

5.5.1 The Roots of Complex Number

If n is a positive integer, using De Moivre's theorem we have,

$$\begin{aligned} z^{1/n} &= \{r(\cos\theta + i\sin\theta)\}^{1/n} \\ &= r^{1/n}\left\{\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right\}. \qquad [0 \le k < n] \end{aligned}$$

Problem 5.2 Evaluate $(-1+i)^{\frac{1}{3}}$.

Solution

$$-1+i = \sqrt{1+1}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right) = \sqrt{2}\left(\cos\left(\frac{3\pi}{4} + 2k\pi\right) + i\sin\left(\frac{3\pi}{4} + 2k\pi\right)\right)$$

Then

$$(-1+i)^{1/3} = \left(\sqrt{2}\right)^{1/3}\left(\cos\left(\frac{3\pi+8k\pi}{4\cdot 3}\right) + i\sin\left(\frac{3\pi+8k\pi}{4\cdot 3}\right)\right).$$

For $k=0$,

$$\sqrt[6]{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right),$$

For $k=1$,

$$\sqrt[6]{2}\left(\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right),$$

For $k = 2$,

$$\sqrt[6]{2}\left(\cos\left(\frac{19\pi}{12}\right) + i\sin\left(\frac{19\pi}{12}\right)\right),$$

Proposition 5.3

$$z^{-1} = \frac{\bar{z}}{|z|^2}. \tag{5.3}$$

Proof Let $z = a + ib$ then

$$z^{-1} = \frac{1}{a+ib} = \frac{1}{a+ib}\frac{a-ib}{a-ib} = \frac{a-ib}{a^2-i^2b^2} = \frac{a-ib}{a^2+b^2} = \frac{\bar{z}}{|z|^2} = \frac{\bar{z}}{z\bar{z}}.$$

Problem 5.3 Evaluate $\left(\sqrt{3}-i\right)^{-\frac{1}{3}}$

Solution *Let* $z = \sqrt{3} - i$, *then*

$$\begin{aligned}
z^{-1} &= \frac{1}{\sqrt{3}-i}\\
&= \frac{1}{\sqrt{3}-i}\frac{\sqrt{3}+i}{\sqrt{3}+i}\\
&= \frac{\sqrt{3}+i}{3+1}\\
&= \frac{\sqrt{3}+i}{4}\\
&= \frac{\sqrt{4}}{4}(\cos 30^\circ + i\sin 30^\circ)\\
&= \frac{1}{2}(\cos(30^\circ + k\cdot 360^\circ) + i\sin(30^\circ + k\cdot 360^\circ))\\
\implies z^{-1/3} &= \frac{1}{\sqrt[3]{2}}(\cos(30^\circ + k\cdot 360^\circ) + i\sin(30^\circ + k\cdot 360^\circ))^{1/3}\\
&= \frac{1}{\sqrt[3]{2}}(\cos(10^\circ + k\cdot 120^\circ) + i\sin(10^\circ + k\cdot 120^\circ))
\end{aligned}$$

For $k = 0$,

$$z^{-1/3} = \frac{1}{\sqrt[3]{2}}(\cos(10^\circ) + i\sin(10^\circ))$$

For $k = 1$,

$$z^{-1/3} = \frac{1}{\sqrt[3]{2}}(\cos(130^\circ) + i\sin(130^\circ))$$

For $k = 2$,

$$z^{-1/3} = \frac{1}{\sqrt[3]{2}}(\cos(250^\circ) + i\sin(250^\circ))$$

Alternate solution

Solution *Let* $z = \sqrt{3} - i$, *then* $r = \sqrt{3+1} = 2$, *and* $\theta = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = \tan^{-1}\tan(-30^\circ) =$

$\tan^{-1}\tan(330°) = 330°$.

$$\begin{aligned}
\sqrt{3} - i &= 2\left(\cos(330°) + i\sin(330°)\right) \\
\implies \left(\sqrt{3} - i\right)^{-1/3} &= \left(2\left(\cos(330°) + i\sin(330°)\right)\right)^{-1/3} \\
&= \frac{1}{\sqrt[3]{2}}\left(\cos(330° + k\cdot 360°) + i\sin(330° + k\cdot 360°)\right)^{-1/3} \qquad [0 \le k < 3] \\
&= \frac{1}{\sqrt[3]{2}}\left(\cos(-110° - k\cdot 120°) + i\sin(-110° - k\cdot 120°)\right)
\end{aligned}$$

For $k = 0$,

$$z^{-1/3} = \frac{1}{\sqrt[3]{2}}\left(\cos(-110°) + i\sin(-110°)\right) = \frac{1}{\sqrt[3]{2}}\left(\cos(250°) + i\sin(250°)\right)$$

For $k = 1$,

$$z^{-1/3} = \frac{1}{\sqrt[3]{2}}\left(\cos(-230°) + i\sin(-230°)\right) = \frac{1}{\sqrt[3]{2}}\left(\cos(130°) + i\sin(130°)\right)$$

For $k = 2$,

$$z^{-1/3} = \frac{1}{\sqrt[3]{2}}\left(\cos(-350°) + i\sin(-350°)\right) = \frac{1}{\sqrt[3]{2}}\left(\cos(10°) + i\sin(10°)\right)$$

Chapter 5 Exercise

1. Define the following
   (a). Complex number
   (b). Absolute value or modulus a complex number
   (c). Argument of a complex number
   (d). Conjugate of a complex number
   (e). Ordered pair of a complex number
   (f). Product of two complex number
2. Write Euler's formula for complex number.
3. Find the polar form of the following
   (a). $-1 + i\sqrt{3}$
   (b). $-5 + i5$
4. Evaluate following
   (a). $\sqrt{i}$
   (b). $(i+1)^4$
   (c). $(-1+i)^{\frac{1}{3}}$
   (d). $(1-i)^{\frac{1}{3}}$
   (e). $(3-i)^{-\frac{1}{3}}$
5. If z_1, z_2 are two complex number then show that $|z_1 z_2| = |z_1||z_2|$.

Chapter 6 Analytic Functions

Introduction

- *Function of Complex variable*
- *Limits*
- *Continuity*
- *Differentiation*
- *Cauchy-Reimann*
- *Harmonic Function*

6.1 Function of Complex Variables

Definition 6.1 (Function)

Let S be a set of complex numbers. A function *f defined on S is a rule that assigns to each z in S a complex number w. The number w is called the value of f at z and is denoted by $f(z)$; that is, $w = f(z)$. The set S is called the domain of definition of f.* ♣

Definition 6.2 (Single Valued Function)

The function $w = f(z)$ is called a single valued function *if for every value of z there is only one value of w.* ♣

Definition 6.3 (Multi Valued Function)

The function $w = f(z)$ is called a multi valued function *if for every value of z there are more than one value of w.* ♣

Example 6.1 The function $w = z^2$ is a single-valued function of z. On the other hand, if $w = z^{\frac{1}{2}}$, then to each value of z there are two values of w. Hence, the function

$$w = z^{\frac{1}{2}}$$

is a multiple-valued (in this case two-valued) function of z.

Definition 6.4 (Polynomial)

For $a_n, a_{n-1}, \ldots, a_0$ complex constants we define $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ is a polynomial of degree n, *where $a_n \neq 0$ and n is a positive integer called the* degree of the polynomial $p(z)$. ♣

Definition 6.5 (Rational Function)

If $P(z)$, and $Q(z)$ are two polynomials then $\frac{P(z)}{Q(z)}$ is called a rational function, *which are defined at each point z except where $Q(z) = 0$.* ♣

6.2 Limits

Definition 6.6

The function $f(z)$ defined in some neighborhood of z_0 is said to have a limit *w_0 at z_0 if, for every given $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$|f(z) - w_0| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta.$$

Mathematically,

$$\lim_{z \to z_0} f(z) = w_0.$$

♣

6.3 Continuity

Definition 6.7 (Continuity)

A complex valued function $f(z)$ is said to be continuous at a point z_0 if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z) - f(z_0)| < \epsilon, \text{ whenever } |z - z_0| < \delta,$$

Mathematically,

$$\lim_{z \to z_0} f(z) = f(z_0).$$

♣

Definition 6.8 (Uniform Continuity)

A function $f(z)$ is said to be uniformly continuous on a set S if, for given $\epsilon > 0$ there exist a $\delta > 0$ such that

$$|f(z_1) - f(z_2)| < \epsilon, \text{ whenever } |z_1 - z_2| < \delta;\ \forall z_1, z_2 \in S.$$

Here $\delta = \delta(\epsilon)$ and δ is independent of z_1 and z_2 in S.

♣

6.4 Derivatives

Definition 6.9 (Derivative)

A complex function $f(z)$ is said to be differentiable at z_0, if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists and finite. This limit is denoted by $f'(z_0)$ and is called the derivative of $f(z)$ at z_0.

♣

Definition 6.10 (Analytic Function)

A complex function $f(z)$ is said to be analytic at a point z_0, if its derivative exist for all z such that $|z - z_0| < \delta$, for some $\delta > 0$, and is said to be analytic in a region R if it is analytic at each point of R. ♣

6.4.1 Cauchy-Riemann Equations

Theorem 6.1 (Cauchy-Riemann Equations)

A necessary condition that $w = f(z) = u(x, y) + iv(x, y)$ be analytic in a region R is that, in R, u and v satisfy the Cauchy –Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{6.1}$$

If the partial derivatives in (6.1) are continuous in R, then the Cauchy –Riemann equations are sufficient conditions that $f(z)$ be analytic in R. ♡

6.5 Harmonic Functions

Definition 6.11 (Harmonic Function)

A real valued function $u(x, y)$ is said to be harmonic *in a region R, if it satisfies the Laplace equation, i.e.*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

♣

Definition 6.12 (Harmonic Conjugate)

The function v is said to be a harmonic conjugate *of u if u and v are harmonic and satisfies the Cauchy-Riemann equations.* ♣

Theorem 6.2

The real and imaginary parts of an analytic function are harmonic function. ♡

Problem 6.1 Find the harmonic conjugate of the function $u = e^{x^2-y^2}\cos(2xy)$ and the corresponding analytic function $f(z) = u + iv$.

Solution *Given that*

$$u = e^{x^2-y^2}\cos(2xy)$$

then, we have,

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2xe^{x^2-y^2}\cos(2xy) - 2ye^{x^2-y^2}\sin(2xy) \\ &= 2e^{x^2-y^2}\left(x\cos(2xy) - y\sin(2xy)\right) \end{aligned} \tag{6.2}$$

$$\begin{aligned}
\frac{\partial u}{\partial y} &= -2ye^{x^2-y^2}\cos(2xy) - 2xe^{x^2-y^2}\sin(2xy) \\
&= -2e^{x^2-y^2}\left(y\cos(2xy) + x\sin(2xy)\right)
\end{aligned} \tag{6.3}$$

Now putting $x = z$ and $y = 0$ in (6.2)-(6.3), *we get*

$$\begin{aligned}
\frac{\partial u}{\partial x} &= 2e^{z^2}(z-0) = 2ze^{z^2} \\
\frac{\partial u}{\partial y} &= 2e^{z^2}(0-0) = 0
\end{aligned}$$

By Milan's theorem we have

$$\begin{aligned}
f'(z) &= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = 2ze^{z^2} \\
\implies f(z) &= \int 2ze^{z^2}dz = \int e^{z^2}d(z^2) = e^{z^2} + c \\
\implies u + iv &= e^{(x+iy)^2} + c \\
&= e^{x^2-y^2}e^{i2xy} + c \\
&= e^{x^2-y^2}\left(\cos(2xy) + i\sin(2xy)\right) + c_1 + ic_2 \quad [let\ c = c_1 + ic_2]
\end{aligned}$$

Equating imaginary parts we have,

$$v = e^{x^2-y^2}\sin(2xy) + c_2$$

and also

$$f(z) = e^{x^2-y^2}\left(\cos(2xy) + i\sin(2xy)\right) + c.$$

Problem 6.2 In aerodynamics and fluid mechanics, the function ϕ and ψ in $f(z) = \phi + i\psi$, where $f(z)$ is analytic, are called the velocity potential and stream function respectively. If $\phi = x^2 + 4x - y^2 + 2y$,

1. find ψ
2. find $f(z)$.

Solution *Given that*

$$\phi = x^2 + 4x - y^2 + 2y$$

then, we have,

$$\frac{\partial \phi}{\partial x} = 2x + 4 \tag{6.4}$$

$$\frac{\partial \phi}{\partial y} = -2y + 2 \tag{6.5}$$

Now putting $x = z$ and $y = 0$ in (6.2)-(6.3), *we get*

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= 2z + 4 \\
\frac{\partial \phi}{\partial y} &= 2
\end{aligned}$$

By Milan's theorem we have

$$\begin{aligned} f'(z) &= \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y} = 2z + 4 - i2 \\ \implies f(z) &= \int (2z + 4 - i2)\, dz = z^2 + 4z - 2zi + c \\ \implies \phi + i\psi &= (x + iy)^2 + 4(x + iy) - 2i(x + iy) + c \\ &= (x^2 - y^2 + 4x + 2y) + i(2xy - 2x + 4y) + c_1 + ic_2 \quad [\textit{let } c = c_1 + ic_2] \end{aligned}$$

Equating imaginary parts we have,

$$\psi = (2xy - 2x + 4y) + c_2$$

and also

$$f(z) = (x^2 - y^2 + 4x + 2y) + i(2xy - 2x + 4y) + c.$$

Chapter 6 Exercise

1. Define the following
   (a). Continuity for function of a complex variable
   (b). Harmonic function
   (c). Analytic function
2. State the theorem Cauchy-Riemann equations.
3. Find the harmonic conjugate of the function $u = e^{x^2-y^2}\cos(2xy)$ and the corresponding analytic function $f(z) = u + iv$.
4. In aerodynamics and fluid mechanics, the function ϕ and ψ in $f(z) = \phi + i\psi$, where $f(z)$ is analytic, are called the velocity potential and stream function respectively. If $\phi = x^2 + 4x - y^2 + 2y$,
   (a). find ψ
   (b). find $f(z)$.

Chapter 7 Complex Integration and Cauchy's Integral Formulas

Introduction

- *Line Integral of a Complex Function*
- *Cauchy's Integral Formula*

7.1 Some Definition

Definition 7.1 (Closed Curve)

If the starting and ending points of a curve coincide then the curve is called a closed curve. ♣

Definition 7.2 (Simple closed curve)

A closed curve which does not intersect itself anywhere is called a simple closed curve ♣

Definition 7.3 (Simply connected region)

A region R is called simply connected *if any any simple closed curve which lies in R can be shrunk to a point without leaving R.* ♣

Definition 7.4 (Multiply connected region)

A region R which is not simply connected is called multiply connected. ♣

Definition 7.5 (Contour)

A contour *is either a single point z_0 or a finite sequence of directed smooth curves $(\gamma_1, \gamma_2, \dots, \gamma_n)$ such that the terminal point of γ_k coincides with the initial point of γ_{k+1} for each $k = 1, 2, \dots, n-1$. In this case one can write $\Gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$. If the terminal point of γ_n coincides with initial point of γ_1 then the contour is said to be the* closed contour. ♣

Note

1. *A single directed smooth curve is a contour with $n = 1$.*
2. *In the case of closed contour the integral is written as $\int_C f(z)dz$ or $\oint_C f(z)dz$.*

7.2 Line Integral of a Complex Function

Theorem 7.1

If $f(z)$ is analytic in a region R and on its closed boundary C, with derivative $f'(z)$ which is continuous at all points inside R and on C then

$$\oint_C f(z)dz = 0.$$

♡

Theorem 7.2

Let $f(z)$ be analytic in a region bounded by two simple closed curves C_1 and C_2 (C_2 lies inside C_1) and on these curves then

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$

♡

Theorem 7.3 (Cauchy's integral formula)

Let $f(z)$ be analytic inside and on a simple closed curve C. If a is any point in C, then

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz,$$

or

$$\oint_C \frac{f(z)}{z-a}dz = 2\pi i f(a),$$

where C is traversed in the positive sense.

♡

Proof We know that if $f(z)$ is analytic in a region bounded by two simple closed curves C and C_1 (C_1 lies inside C) and on these curves then

$$\oint_{C_1} f(z)dz = \oint_C f(z)dz \tag{7.1}$$

Here the function $\frac{f(z)}{z-a}$ is analytic inside and on C except at the point $z = a$.

$$\oint_C \frac{f(z)}{z-a}dz = \oint_\Gamma \frac{f(z)}{z-a}dz \qquad [\text{ By } (7.1)] \tag{7.2}$$

where Γ is a circle with center a and radius r. We have

$$\begin{aligned} |z-a| &= r \\ \implies z-a &= re^{i\theta}, \qquad [0 \le \theta \le 2\pi] \\ \implies z &= a + re^{i\theta}, \\ \implies dz &= 0 + ire^{i\theta}d\theta = ire^{i\theta}d\theta \end{aligned}$$

Putting these values in (7.2), we get

$$\begin{aligned} \oint_C \frac{f(z)}{z-a}dz &= \int_0^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}} ire^{i\theta}d\theta \\ &= i\int_0^{2\pi} f(a+re^{i\theta})d\theta \end{aligned}$$

Taking limit $r \to 0$ on both sides and making use of the continuity of $f(z)$, we get

$$\begin{aligned}
\oint_C \frac{f(z)}{z-a}dz &= \lim_{r\to 0} i\int_0^{2\pi} f(a+re^{i\theta})d\theta \\
&= if(a)\int_0^{2\pi} d\theta = if(a)[\theta]_0^{2\pi} = i2\pi f(a) \\
\implies f(a) &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz \qquad (7.3)
\end{aligned}$$

Theorem 7.4 (Liouville's theorem)

If for all z in the entire complex plane, $f(z)$ is analytic and bounded, then $f(z)$ must be a constant.

Proof Let a and b any two points in the z plane. Suppose that C is any circle of radius r and center at a, containing a point b. Then by Cauchy's integral formula, we have

$$\begin{aligned}
f(b)-f(a) &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-b}dz - \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz \\
&= \frac{1}{2\pi i}\oint_C f(z)\left(\frac{1}{z-b}-\frac{1}{z-a}\right)dz \\
&= \frac{1}{2\pi i}\oint_C f(z)\left(\frac{z-a-z+b}{(z-a)(z-b)}\right)dz \\
&= \frac{b-a}{2\pi i}\oint_C \frac{f(z)}{(z-a)(z-b)}dz \qquad (7.4)
\end{aligned}$$

Now, $f(z)$ is bounded, so there exist a constant M such that $|f(z)| \le M$. Also we have, $|z-a| = r$,

$$|z-b| = |z-a+a-b| \ge |z-a| - |a-b| = r - |a-b|$$

If we choose r so large such that $|a-b| < r/2$. Then we have,

$$|z-b| \ge r - r/2 = r/2.$$

Again length of the circle is $\oint_C dz = 2\pi r$. Now from (7.4) we get,

$$\begin{aligned}
|f(b)-f(a)| &= \left|\frac{b-a}{2\pi i}\oint_C \frac{f(z)}{(z-a)(z-b)}dz\right| \\
&\le \frac{|b-a|}{2\pi}\frac{|f(z)|}{|z-a||z-b|}\left|\oint_C dz\right| \\
&\le \frac{|b-a|}{2\pi}\frac{M}{r\cdot r/2}2\pi r \\
&= \frac{2|b-a|M}{r}
\end{aligned}$$

When $r \to \infty$ then

$$\begin{aligned}
|f(a)-f(b)| &= 0 \\
\implies f(a)-f(b) &= 0 \\
\implies f(a) &= f(b).
\end{aligned}$$

for any arbitrary points a and b in the complex plane.

Chapter 7 Exercise

1. State and prove Cauchy's integral formula.
2. State and prove Liouville's theorem.

Chapter 8 Singularities, Residue and Some Theorem

Introduction

- *Zero or root of an analytic function*
- *Singularity*
- *Poles*
- *Taylor's theorem*
- *Laurent's series*
- *Cauchy's Residue theorem*

8.1 Zero or root of Analytic Function

Definition 8.1 (Root)

A value of z for which the analytic function $f(z) = 0$ is called a zero *or* root *of $f(z)$. If $f(z) = (z - z_0)^n g(z)$, where $g(z)$ is analytic and $g(z) \neq 0$ and n is a positive integer, then $z = z_0$ is called a* zero *or* root *of order n of the function $f(z)$.* ♣

Definition 8.2 (Simple zero)

If $f(z)$ has a zero of order one at $z = z_0$, then $f(z)$ is said to have a simple zero *at $z = z_0$.* ♣

8.2 Singularity

Definition 8.3 (Singular point or Critical point)

A point at which an analytic function $f(z)$ fails to be analytic is called a singular point. ♣

Definition 8.4

If $f(z)$ is analytic everywhere in some region except at an interior point $z = a$, we call $z = a$ an isolated singularity *of $f(z)$. If the point $z = z_0$ is not an isolated singularity then it is called a* non-isolated singularity. ♣

Definition 8.5 (Pole)

If $f(z) = \frac{\phi(z)}{(z-a)^n}$, $\phi(a) \neq 0$, where $\phi(z)$ is analytic everywhere in a region including $z = a$, and if n is a positive integer, then $f(z)$ has an isolated singularity at $z = a$ which is called a pole of order n. If $n = 1$, the pole is often called a simple pole; if $n = 2$ it is called a double pole, etc. ♣

Definition 8.6 (Removable Singularity)

If $\lim_{z \to z_0} f(z)$ *exists then* z_0 *is called a removable singualrity of* $f(z)$*.* ♣

Definition 8.7 (Essential singularity)

A singular point which is not a pole, branch point or removable point is called essential singularity. ♣

Definition 8.8 (Singularity at infinity)

The function $f(z)$ *has a singularity at* $z = \infty$ *if* $w = 0$ *is a singularity of* $f(\frac{1}{w})$*.* ♣

Definition 8.9 (Entire function)

A function that is analytic everywhere in the finite plane [i.e., everywhere except at ∞*] is called an* entire function *or* integral function. ♣

The functions e^z, $\sin z$, $\cos z$ are entire functions.

Definition 8.10 (Meromorphic function)

A function that is analytic everywhere in the finite plane except at a finite number of poles is called a meromorphic function. ♣

8.2.1 Rules for poles and singularities

1. - If $\lim_{z \to z_0} f(z) = \infty$ then $z = z_0$ is a pole of $f(z)$.
   - If there are only m terms in the negative powers of $z - z_0$ then $z = z_0$ is a pole of order m.
2. If $\lim_{z \to z_0}$ exists finitely then $z = z_0$ is a removable singularity.
3. If $\lim_{z \to z_0}$ does not exist then $z = z_0$ is an essential singularity.
4. If the principal part of $f(z)$ contains infinite numbers of terms then $z = z_0$ is an isolated essential singularity.

Problem 8.1 Locate in the finite z plane all the singularities, if any, of each function and name them.

1. $\frac{z^2}{(z+1)^3}$,
2. $\frac{1-\cos(z)}{z}$.

Solution

1. $\frac{z^2}{(z+1)^3}$. $z - 1$ *is a pole of order 3.*
2. $\frac{1-\cos(z)}{z}$. $z = 0$ *is appears to be a singularity. However, since* $\lim_{z \to z_0} \frac{1-\cos(z)}{z} = 0$, *it is a removable singularity.*

8.3 Taylor's theorem

Theorem 8.1 (Taylor's Theorem)

If $f(z)$ is analytic for all values of z inside a circle C with center at a,

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \frac{(z-a)^3}{3!}f'''(a) + \dots$$

♡

8.4 Laurent's theorem

Theorem 8.2 (Laurent's Theorem)

If $f(z)$ is analytic inside and on the boundary of the ring shaped region R bounded by two concentric circles C_1 and C_2 with center at a and radii r_1 and r_2 respectively $(r_2 < r_1)$ then for all z in R,

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-a)^n} \tag{8.1}$$

where

$$\begin{aligned} a_n &= \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^{n+1}} \qquad n = 0, 1, 2, 3, \dots \\ a_{-n} &= \frac{1}{2\pi i}\oint_{C_2} \frac{f(w)}{(w-a)^{-n+1}} \qquad n = 1, 2, 3, \dots \end{aligned} \tag{8.2}$$

♡

8.5 Residues and Residues Theorem

Definition 8.11 (Residues)

If the function $f(z)$ is analytic within a circle C of radius r and center a, except at $z = a$, then the coefficient a_{-1} of $\frac{1}{z-a}$ in the Laurent's expansion [see (8.2)*] around z is called the* residue *of $f(z)$ at $z = a$. It is denoted by Res(a) or a_{-1}.*

♣

Problem 8.2 Find the Laurent series for $\frac{z}{(z+1)(z+2)}$; at $z = -1$, name the singularity and give the region of convergence of the series.

Solution *Let* $z+1=u$. *Then*

$$\begin{aligned}\frac{z}{(z+1)(z+2)} &= \frac{u-1}{u(u+1)} = \frac{u-1}{u}\left(1-u+u^2-u^3+u^4-\dots\right) \\ &= (u-1)\left(\frac{1}{u}-1+u-u^2+u^3-\dots\right) \\ &= \left(1-u+u^2-u^3+u^4-\dots\right)-\left(\frac{1}{u}-1+u-u^2+u^3-\dots\right) \\ &= -\frac{1}{u}+2-2u+2u^2-2u^3+\dots \\ &= -\frac{1}{z+1}+2-2(z+1)+2(z+1)^2-2(z+1)^3+\dots\end{aligned}$$

$z=-1$ *is a pole of order 1, or simple pole. The series convergence for all values of* z *such that* $0<|z+1|<1$.

Problem 8.3 Determined the residues of

$$\frac{z^2}{(z-2)(z^2+1)}$$

at all poles.

Solution $z=2,i,-i$, *are three simple poles of* $\frac{z^2}{(z-2)(z^2+1)}$. *Then*

Residue at $z=2$ *is*

$$\lim_{z\to 2}(z-2)\left(\frac{z^2}{(z-2)(z^2+1)}\right)=\frac{4}{5}.$$

Residue at $z=i$ *is*

$$\lim_{z\to i}(z-i)\left(\frac{z^2}{(z-2)(z-i)(z+i)}\right)=\frac{i^2}{(i-2)2i}=\frac{i(-i-2)}{2\sqrt{1+2^2}}=\frac{1-2i}{10}.[see(5.3)]$$

Residue at $z=-i$ *is*

$$\lim_{z\to -i}(z+i)\left(\frac{z^2}{(z-2)(z-i)(z+i)}\right)=\frac{i^2}{(-i-2)(-2i)}=\frac{i}{(2+i)(2)}=\frac{i(2-i)}{2\sqrt{1+2^2}}=\frac{1+2i}{10}.$$

Theorem 8.3 (Cauchy's Residue Theorem)

If $f(z)$ *is analytic inside and on a simple closed curve* C *except at a finite number of points* a, b, $c,\dots$ *inside* C *at which the residues are* $a_{-1},b_{-1},c_{-1},\dots$ *respectively, then*

$$\oint_C f(z)dz = 2\pi i\left(a_{-1}+b_{-1}+c_{-1}+\dots\right)=2\pi i(\textit{Sum of residues})$$

♡

8.6 Evaluation of Definite Integral

Problem 8.4 Evaluate

$$\oint_C \frac{e^z dz}{(z-1)(z+3)^2},$$

where C is given by $|z|=3/2$.

Solution *Since* $|z|=3/2$ *encloses only the simple pole at* $z=1$, *residue at simple pole* $z=1$ *is*

$$\lim_{z\to 1}\left((z-1)\frac{e^z}{(z-1)(z+3)^2}\right)=\lim_{z\to 1}\frac{e^z}{(z+3)^2}=\frac{e}{16}.$$

The required integral $= 2\pi i\left(\frac{e}{16}\right) = \frac{i\pi e}{8}$.

Problem 8.5 Show that

$$\int_0^{2\pi} \frac{d\theta}{5+3\sin\theta} = \frac{\pi}{2}.$$

Solution *Let* $z = e^{i\theta}$ *then* $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-z^{-1}}{2i}$ *and* $dz = ie^{i\theta}d\theta = izd\theta$. *Then we have*

$$\begin{aligned}\int_0^{2\pi} \frac{d\theta}{5+3\sin\theta} &= \oint_C \frac{1}{5+3\left(\frac{z-z^{-1}}{2i}\right)}\frac{dz}{iz}\\ &= \oint_C \frac{1}{5+3\left(\frac{z-z^{-1}}{2i}\right)}\frac{dz}{iz}\\ &= \oint_C \frac{2}{3z^2+10iz-3}dz\end{aligned}$$

Where C *is the circle of unite radius with center at the origin. The poles of* $\frac{2}{3z^2+10iz-3}$ *are the simple poles*

$$z = \frac{-10i \pm \sqrt{-100+36}}{6} = \frac{-10i \pm 8i}{6} = -3i, -i/3.$$

only $-i/3$ *lies inside* C.

Residue at $-i/3$ *is*

$$\lim_{z\to -i/3}\left(z+\frac{i}{3}\right)\left(\frac{2}{3z^2+10iz-3}\right) = \lim_{z\to -i/3}\frac{2}{3(z+3i)} = \frac{2}{3(-i/3+3i)} = \frac{2}{8i} = \frac{1}{4i}.$$

Then

$$\int_0^{2\pi} \frac{d\theta}{5+3\sin\theta} = 2\pi i\left(\frac{1}{4i}\right) = \frac{\pi}{2}.$$

Problem 8.6 Show that

$$\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}d\theta = \frac{\pi}{12}.$$

Solution

$$\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}d\theta \tag{8.3}$$

Let $z = e^{i\theta}$ *then* $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2}$. *We also have*

$$\cos 3\theta = \frac{e^{3i\theta}+e^{-3i\theta}}{2} = \frac{z^3+z^{-3}}{2}$$

and $dz = izd\theta$. Then

$$\begin{aligned}
\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta &= \oint_C \frac{\frac{z^3+z^{-3}}{2}}{5-4\left(\frac{z+z^{-1}}{2}\right)} \frac{dz}{iz} \\
&= \frac{1}{2i}\oint_C \frac{z^{-3}(z^6+1)}{5z-(2z^2+2)} dz \\
&= -\frac{1}{2i}\oint_C \frac{z^6+1}{z^3\left(2z^2-5z+2\right)} dz \\
&= -\frac{1}{2i}\oint_C \frac{z^6+1}{z^3\left(2z-1\right)\left(z-2\right)} dz
\end{aligned}$$

Where C is the circle of unite radius with center at the origin. The integrand has a pole of order 3 at $z=0$ and a simple pole $z=\frac{1}{2}$ within C.

Residue at $z=0$ is

$$\begin{aligned}
&\lim_{z\to 0} \frac{1}{2!}\frac{d^2}{dz^2}\left(z^3 \frac{z^6+1}{z^3\left(2z-1\right)\left(z-2\right)}\right) \\
= &\lim_{z\to 0} \frac{1}{2!}\frac{d^2}{dz^2}\left(\frac{z^6+1}{\left(2z-1\right)\left(z-2\right)}\right) \qquad (8.4)
\end{aligned}$$

Let $u = z^6+1$ and $v = 2z^2-5z+2 = (2z-1)(z-2)$then we have

$$u(0)=1; \quad u'(z)=6z^5; \quad u'(0)=0; \quad u''(z)=30z^4; \quad u''(0)=0.$$
$$v(0)=2; \quad v'(z)=4z-5; \quad v'(0)=-5; \quad v''(z)=4; \quad v''(0)=4.$$

Also we have,

$$\begin{aligned}
\left(\frac{u}{v}\right)' &= \left(\frac{u'v-uv'}{v^2}\right) \\
\implies \left(\frac{u}{v}\right)'' &= \left(\frac{u'v-uv'}{v^2}\right)' \\
&= \frac{\left(u''v-uv''\right)v^2 - 2\left(u'v-uv'\right)vv'}{v^4} \\
&= \frac{u''v-uv''}{v^2} - \frac{2\left(u'v-uv'\right)v'}{v^3} \qquad (8.5)
\end{aligned}$$

Now we can find

$$\left(\frac{u}{v}\right)''(0) = \frac{0\cdot 2-1\cdot 4}{4} - \frac{2\left(0\cdot 2-1\cdot(-5)\right)(-5)}{8} = \frac{-4+25}{4} = \frac{21}{4}$$

Using this value in (8.4) we have residue at $z=0$ is $\frac{21}{2\cdot 4} = \frac{21}{8}$.

Residue at $z=1/2$ is

$$\begin{aligned}
&\lim_{z\to 1/2}(z-\frac{1}{2})\frac{z^6+1}{z^3\left(2z-1\right)\left(z-2\right)} \\
= &\lim_{z\to 1/2}\frac{z^6+1}{2z^3\left(z-2\right)} \\
= &\frac{(1/2)^6+1}{2(1/2)^3\left((1/2)-2\right)} \\
= &-\frac{65/64}{3/8} = -\frac{65}{24}
\end{aligned}$$

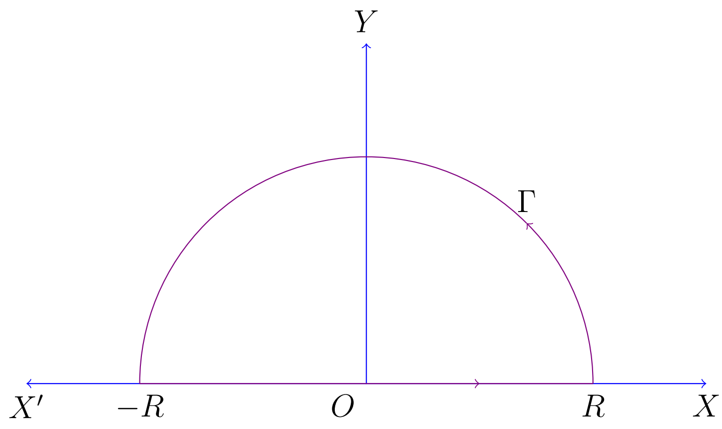
Then

$$-\frac{1}{2i}\oint_C \frac{z^6+1}{z^3(2z-1)(z-2)}dz = -\frac{1}{2i}(2\pi i)\left(\frac{21}{8}-\frac{65}{24}\right) = \frac{\pi}{12}.$$

Problem 8.7 Evaluate

$$\int_0^\infty \frac{dx}{x^4+1}.$$

Solution *Consider $\oint_C \frac{dz}{z^4+1}$, where C is the closed contour consisting of the line from $-R$ to R and*



the semi-circle Γ, traverse clockwise.

Since $z^4+1=0$, when $z=e^{i\pi/4}$, $e^{i3\pi/4}$, $e^{i5\pi/4}$, $e^{i7\pi/4}$, these are simple poles of $1/(z^4+1)$. Only the poles $e^{i\pi/4}$ and $e^{i3\pi/4}$ lie within C. Then L'Hospital's rule,

Residue at $e^{i\pi/4}$ is

$$\lim_{z\to e^{i\pi/4}}\left[z-e^{i\pi/4}\frac{1}{z^4+1}\right] = \lim_{z\to e^{i\pi/4}}\frac{1}{4z^3} = \frac{1}{4}e^{-i3\pi/4}.$$

Residue at $e^{i3\pi/4}$ is

$$\lim_{z\to e^{i3\pi/4}}\left[z-e^{i3\pi/4}\frac{1}{z^4+1}\right] = \lim_{z\to e^{i3\pi/4}}\frac{1}{4z^3} = \frac{1}{4}e^{-i9\pi/4}.$$

Thus

$$\begin{aligned}
\oint_C \frac{dz}{z^4+1} &= 2i\pi\left(\frac{1}{4}e^{-i3\pi/4}+\frac{1}{4}e^{-i9\pi/4}\right)\\
&= \frac{i\pi}{2}\left(\cos\left(\frac{3\pi}{4}\right)-i\sin\left(\frac{3\pi}{4}\right)+\cos\left(\frac{9\pi}{4}\right)-i\sin\left(\frac{9\pi}{4}\right)\right)\\
&= \frac{i\pi}{2}\left(-\cos\left(\frac{\pi}{4}\right)-i\sin\left(\frac{\pi}{4}\right)+\cos\left(\frac{\pi}{4}\right)-i\sin\left(\frac{\pi}{4}\right)\right)\\
&= \frac{-i^2\pi}{2}\left(2\sin\left(\frac{\pi}{4}\right)\right) = \frac{\pi\sqrt{2}}{2}\\
\implies \int_{-R}^{R}\frac{dx}{x^4+1}+\int_\Gamma \frac{dz}{z^4+1} &= \frac{\pi\sqrt{2}}{2}
\end{aligned}$$

Now taking the limit on both side

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{x^4+1}+\lim_{R\to\infty}\int_\Gamma\frac{dz}{z^4+1} = \frac{\pi\sqrt{2}}{2} \tag{8.6}$$

Let $z = Re^{i\theta}$, *and* $R \to \infty$ *then*

$$\begin{aligned}
& \left|\frac{1}{R^4 e^{i4\theta}+1}\right| \leq \frac{1}{|R^4 e^{i4\theta}|-1} = \frac{1}{R^4-1} \leq \frac{2}{R^4} \\
\Longrightarrow \quad & \lim_{R\to\infty}\left|\frac{1}{z^4+1}\right| = \lim_{R\to\infty}\left|\frac{1}{R^4 e^{i4\theta}+1}\right| = 0 \\
\Longrightarrow \quad & \lim_{R\to\infty}\int_\Gamma \frac{dz}{z^4+1} = 0. \qquad (8.7)
\end{aligned}$$

Using (8.7) *in* (8.6) *we have,*

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{dx}{x^4+1} &= \frac{\pi\sqrt{2}}{2} \\
2\int_{0}^{\infty}\frac{dx}{x^4+1} &= \frac{\pi\sqrt{2}}{2} \\
\int_{0}^{\infty}\frac{dx}{x^4+1} &= \frac{\pi\sqrt{2}}{4}
\end{aligned}$$

Chapter 8 Exercise

1. Define Poles with an example.
2. Locate in the finite z plane all the singularities, if any, of each function and name them.
   (a).
   $$\frac{z^2}{(z+1)^3},$$
   (b).
   $$\frac{1-\cos(z)}{z}.$$
3. Define Taylor's series.
4. When a sequence is convergent or divergent?
5. Define Laurent's series?
6. Find the Laurent series about the indicated singularity for $\frac{z}{(z+1)(z+2)}$; at $z=-1$, name the singularity and give the region of convergence of the series.
7. Determined the residues of
   $$\frac{z^2}{(z-2)(z^2+1)}$$
   at $z = 2, i, -i$.
8. Show that
   $$\int_0^{2\pi}\frac{\cos 3\theta}{5-4\cos\theta}\,d\theta = \frac{\pi}{12}.$$
9. Evaluate
   $$\oint_C \frac{e^z dz}{(z-1)(z+3)^2},$$
   where C is given by $|z| = 3/2$.

10. Show that

$$\int_{0}^{2\pi} \frac{d\theta}{5+3\sin\theta} = \frac{\pi}{2}.$$

11. Evaluate

$$\int_0^\infty \frac{dx}{x^4+1}.$$